Final Exam

• Final Exam on
  – 02: Dec 14 Thursday 1:30- 4:00pm at TH331

- Review for Final Exam
  - Covers counting and recursion (100 points)
  - 1 question from each midterm range (45 points)
  - Some notes on Final Exam
  - HW8 & HW9
    - HW9 is not homework. Use questions in this slide.
  - Conclusion of Course

Final Exam

• 6 mandatory questions on counting and recursion (100%)
• 3 optional questions for the entire course (45%)
• One page cheat sheet only. Bring scratch papers and calculator (you will want a calculator).
HW#8 Q2 (2pt)

Use a tree diagram to determine the # of subsets of \{3,7,9,11,24\} with the property that the sum of the elements in the subset is less than 28.

• ANS
HW#8 Q4 (2pt)

How many permutations of \{a,b,c,d,e,f,g\} end with a?

- ANS

\(P(7,7) = 7!\)

\(P(6,6) = 6! = 720\)

HW#8 Q6 (a) (2pt)

A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes are there in total?

- ANS

Each trial has two possible values.

Then you have 10 trials.

Involving the product rule,

\(2^{10} = 1024\).
HW#8 Q6 (b) (2pt)

A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes contain exactly two heads?

\[ \binom{10}{2} = \frac{D(10, 2)}{2!} = \frac{10!}{8!2!} = \frac{10 \times 9}{2} = 45 \]

HW#8 Q6 (c) (2pt)

A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes contain at most three tails?

\[ \text{Sum rule} \]

\[ \binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0} = 176 \]
HW#8 Q6 (d) (2pt)

A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes contain the same # of heads and tails?

• ANS

\[ C(10, 5) = \frac{10!}{5! \cdot 5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2 \cdot 7 \cdot 2 \cdot 7 = 252 \]

HW#8 Q8 (a) (2pt)

How many strings of six lowercase letters from English alphabet contain the letter a?

• ANS

Permutations with Repetition.

\[ \text{All combinations} = \frac{6!}{(6-a)!} \]

\[ = 26 - 25 = 6 - 25 \]

\[ = 6 \cdot 7 \cdot 5 \cdot 151 \]

\[ = \sum_{a=1}^{6} \text{Total} \text{ # \# (No)} \]

\[ = \sum_{a=1}^{6} \text{Total} \]
HW#8 Q8 (b) (2pt)

How many strings of six lowercase letters from English alphabet contain the letters a and b?

- ANS

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

\[ A \cap B = |A_{a\text{ and } b}| - |A_{a\text{ or } b}| \]

\[ |A| = 26^6 \]

\[ |B| = 25^6 \]

\[ |A_{a\text{ and } b}| = 25^5 \times 2 \]

\[ |A_{a\text{ or } b}| = 26 \times 25^5 \]

\[ |A \cup B| = 26^6 - (25^6 \times 2) \]

\[ |A \cap B| = 25^5 \times 2 \times 2 \]

\[ \# = 26^6 - 25^6 \times 2 \times 2 \]

\[ = 26^6 - 25^6 \times 2 \]

\[ = 26^6 - 25^6 \times 2 \times 2 \]

\[ = 26^6 - 25^6 \times 2 \times 2 \]

\[ = 11,737,502 \]


HW#8 Q8 (c) (2pt)

How many strings of six lowercase letters from English alphabet contain the letters a and b in consecutive position with a preceding b, with all the letters distinct?

- ANS

Permutation W/ Repetition

\[ \left\{ \begin{array}{c}
\text{ab} \\
\text{ba} \\
\text{abc} \\
\text{bac} \\
\text{bca}
\end{array} \right. \)

5 possibilities

For each case, we have \( P(24, 4) \) possibilities.

Thus, \( 5 \times P(24, 4) \) possibilities.

\[ \frac{24!}{20!} = \frac{24!}{20!} = 5 \times 24 \times 23 \times 22 \times 21 \]

\[ = 1,275,200 \]
HW#8 Q8 (d) (2pt)

How many strings of six lowercase letters from English alphabet contain the letters a and b where a is somewhere to the left of b in the string, with all the letters distinct?

\[ \text{\textbf{ANS}} \]

\[ \binom{6}{2} \text{ just put 'a' always at left} \]

\[ \binom{6}{2} \cdot P(24,4) \]

\[ = 3,825,360 \]

HW#8 Q10 (2pt)

The row of Pascal’s triangle containing the binomial coefficients \(0 \leq k \leq 10\) is:

\[ 1 \ 10 \ 45 \ 120 \ 210 \ 252 \ 210 \ 120 \ 45 \ 10 \ 1 \]

use Pascal’s identity to produce the row immediately following this row in Pascal’s triangle.

\[ \text{\textbf{ANS}} \]

\[ \binom{n+1}{k} = \binom{n}{k-1} \cdot \binom{n}{k} \]
How many different combinations of pennies, nickels, dimes, quarters and half dollars can a piggy bank contain if it has 20 coins in it?

• ANS  \[ \binom{5}{20} \]

\[ \binom{k+n-1}{n} = \binom{24}{20} = \frac{24!}{4!20!} = 10626 \]
HW#9 Q1 (i)

Give a recursive definition of (i) \( a_n = 1 + (-1)^n \)

- **ANS**

| \( n \) | 1 | 2 | 3 | 4 | 5 | ...
|-------|---|---|---|---|---|---
| \( a_n \) | 0 | 2 | 0 | 2 | 0 | ...

Thus:

\[
\begin{align*}
\{ & a_1 = 0 \\
& a_2 = 2 \\
& a_n = a_{n-2}, \quad n \geq 3
\end{align*}
\]

1) List elements from left to right by increasing \( n \) one-by-one
2) Evaluate
   - i) difference
   - ii) ratio
3) Find a relationship
   - In order to find a constant
4) Derive a general equation
   - \( a_n = 1, a_{n-2}, \ldots \)
5) Solve for \( a_n \)

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HW#9 Q1 (ii)

Give a recursive definition of (i) \( a_n = n^2 \).

- **ANS**

| \( n \) | 1 | 2 | 3 | 4 | 5 | ...
|-------|---|---|---|---|---|---
| \( a_n \) | 1 | 4 | 9 | 16 | 25 | ...

From the definition, we have \( a_{n-1} = (n-1)^2 \)

\[
a_n = n^2 = (n-1)^2 + 2n-1 = a_{n-1} + 2n-1
\]

\[
\begin{align*}
\{ & a_1 = 1 \\
& a_n = a_{n-1} + (2n-1), \quad n \geq 2
\end{align*}
\]
**HW#9 Q4**

Solve the following recurrence relation:

- **ANS**

\[ T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & \text{if } n \geq 2 \end{cases} \]

\[ T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n \]
\[ = 2 \cdot \left(2 \cdot T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + n \]
\[ = 2^2 \cdot \left(2 \cdot T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + 2n \]
\[ = 2^3 \cdot \left(2 \cdot T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + 3n \]
\[ \vdots \]
\[ = 2^k \cdot T\left(\frac{n}{2^k}\right) + kn \]

(by expanding k times)

\[ \text{thus } \frac{n}{2^k} = 1 \implies 2^k = n \implies k = \log_2{n} \]

\[ T(n) = 2^{\log_2{n}} \cdot T(1) + n \log_2{n} = n + n\log_2{n} \]

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**Conclusion**

- You learned the basis for concrete thinking required to be a successful computer scientist/engineer. [Set, Logic, Proof, Relation, Function, Algorithm, Numbers, Graph, Tree, Counting, Recursion]

- Analytical skills in DM allows you to know
  - How to design better algorithm/software
  - Why your algorithm is good/bad (knowledge is power)

- Stay in touch!
  - Biomedical Image Analysis (CSC621)
  - Machine Learning and Artificial Intelligence (CSC872)