Overview

• HW8 due next Tuesday. Work on it!

• Last Lecture
  – Sum Rule: \( A \text{ or } B \text{ or } C \)
  – Product Rule: \( A \text{ and } B \text{ and } C \)
  – Principle of Inclusion-Exclusion, Tree Diagram, Pigeonhole principle
  – Permutations

• This Lecture: Counting!
  – Combination
  – Permutations and combinations with repeats
  – Pascal’s Identity
  – Binomial Coefficients
  – Pascal Triangle

Combinations

• Example: How many ways are there to pick a set of 3 people from a group of 6?

There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so there are \( 6 \cdot 5 \cdot 4 = 120 \) ways to do this.

\[
P(6, 3) = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 6 \cdot 5 \cdot 4
\]

This is not the correct result! \( \binom{6}{3, 3} \neq \binom{6}{3, 3} \)

For example, picking person \( C \), then person \( A \), and then person \( E \) leads to the same group as first picking \( E \), then \( C \), and then \( A \).

However, these cases are counted separately in the above formulation.

\[
\{C, A, E\} = \{E, C, A\}
\]

So how can we compute how many different subsets of people can be picked (i.e., we want to disregard the order of picking)?
• An \( r \)-combination of elements of a set is an unordered selection of \( r \) elements from the set. Thus, an \( r \)-combination is simply a subset of the set with \( r \) elements.

• Example: Let \( S = \{1, 2, 3, 4\} \).
Then \( \{1, 3, 4\} = \{3, 1, 4\} \) is an example member of 3-combination.

• The number of all possible \( r \)-combinations from a set with \( n \) distinct elements is denoted by \( \binom{n}{r} \) (“\( n \) choose \( r \)”).

• Example: \( \binom{4}{2} = 6 \), i.e. the 2-combinations of a set \( \{1, 2, 3, 4\} \) are \( \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \).

• How can we derive \( \binom{n}{r} \)? Formula?
Consider that we can obtain \( \binom{P(n, r)}{r} \) of a set in the following way:
First, find all \( r \)-combinations of the set, i.e. \( \binom{C(n)}{r} \)
Then, for each \( r \)-combination, generate all possible orderings, i.e. \( \binom{P(r, r)}{r} = r! \). Therefore, we have: \( \binom{P(n, r)}{r} = \binom{C(n)}{r} \binom{P(r, r)}{r} \)

\[
\binom{C(n, r)}{r} = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!} \div \frac{r!}{(r-r)!} = \frac{n!}{r!(n-r)!} \div \frac{r!}{r!(r-r)!} = \binom{n}{r} \binom{P(r)}{r}.
\]

• Now we can answer our question: How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?
\( \binom{P(6, 3)}{3} = 6!/3! \cdot 3! = 720/(6 \cdot 3) = 720/18 = 20 \)

• Corollary: Let \( n \) and \( r \) be nonnegative integers with \( r \leq n \).
Then \( \binom{C(n, r)}{r} = \binom{C(n, n-r)}{r} \).

\[
\binom{C(4, 3)}{3} = \binom{C(4, 1)}{3} = \binom{C(4, 7)}{3}.
\]

Note: each choice \( r \)-elements determine a unique choice of \((n-r)\)-elements.
Summary:

\[ C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{P(n, r)}{r!} = \binom{n}{r} \]

Use \( C(n, r) \) = # of combinations of selecting \( r \) distinct objects from \( n \) distinct objects

\[ \frac{n!}{(n-r)!r!} = \frac{P(n, r)}{r!} = \binom{n}{r} \]

Note: \( C(n, 0) = \binom{n}{0} = 1 \) for \( n \geq 4 \)

\( C(n, 1) = \binom{n}{1} = n \)

\( C(n, n) = \binom{n}{n} = 1 \)

\[ C(n, r) = C(n, n-r). \]

• Example: A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

Ans: \( C(8, 6) \times C(7, 5) = \frac{8!}{(6! \cdot 2!)} \times \frac{7!}{(5! \cdot 2!)} = 28 \times 21 = 588 \)

• Example: A committee of 8 students is to be selected from a class consisting of 19 freshmen and 34 sophomores.

In how many ways can 3 freshmen and 5 sophomores be selected?  Ans: \( C(19, 3) \times C(34, 5) \)

In how many ways can a committee with exactly 1 freshman be selected?  Ans: \( C(19, 1) \times C(34, 7) \)

In how many ways can a committee with at most 1 freshman be selected?  0 freshman + 1 freshman = \( C(34, 8) + C(19, 1) \times C(34, 7) \)

In how many ways can a committee with at least 1 freshman be selected?  All 8-combination – no freshman = \( C(53, 8) - C(34, 8) \)
Recall from previous examples:
How many bit strings of length 8 with exactly one 1?
This problem is same as:
how many 1-combination from set \{1,2,3,4,5,6,7,8\} where each element is a position in bit strings?
Answer: $\binom{8}{1} = 8$

How many bit strings of length 8 with exactly two 1s?
Answer: $\binom{8}{2}$

How many bit strings of length 8 with exactly $r$ 1s?
Answer: $\binom{8}{r}$

How many bit strings of length 8 with at most two 1s?
Answer: $\binom{8}{0} + \binom{8}{1} + \binom{8}{2}$

Example: Consider a 5-card hand from a 52-card deck:
How many hands are possible?
Ans: $\binom{52}{5} = 2,598,960$

How many hands consist of all diamonds?
Ans: $\binom{13}{5} = 1,287$

How many hands consist of all the same suit (Flush)?
Ans: $\binom{13}{5} + \binom{13}{5} + \binom{13}{5} + \binom{13}{5} = 5,148$

How many hands contain three of a kind?
\[ \rightarrow \] exactly 3 of a kind + exactly 4 of a kind
Ans: $13 \times \binom{4}{3} \times \binom{48}{2} + 13 \times \binom{4}{4} \times \binom{48}{1} = 58,656 + 624$

How many hands contain a full house?
\[ \rightarrow \] 3 of a kind with a pair
Ans: $13 \times \binom{4}{3} \times 12 \times \binom{4}{2} = 3,744$
• \( P(n, r) \) and \( C(n, r) \) assume that each object will only be selected once without repetition. **Now, suppose \( n \) objects are available for reuse (or with repetition)**

\[
P(n, r) = \frac{n!}{(n-r)!} \quad \text{and} \quad C(n, r) = \frac{n!}{r!(n-r)!}
\]

• **Permutations of \( r \) objects out from \( n \) objects with repetition** is easy:

1st object: \( n \) choices; \( \bigg\rceil \bigg) \)
2nd object: \( n \) choices...
\( r \)th object: \( n \) choices.

Total: \( n^r \)

\[
\begin{align*}
\text{1st} & \quad \text{2nd} & \quad \text{3rd} \\
\bigg\rceil \bigg) & \quad \bigg\rceil \bigg) & \quad \bigg\rceil \bigg)
\end{align*}
\]

Combinations of \( r \) objects out from \( n \) objects with repetition:

- Example: A jeweler is designing a pin with 5 stones chosen from diamonds, rubies, and emeralds. How many ways can the stones be selected? \( (n = 3, r = 5) \)

- I.e. 5-combinations with repetition allowed from three-element set.

\[
\begin{align*}
\bigg\rceil \bigg) & \quad \bigg\rceil \bigg) & \quad \bigg\rceil \bigg) \\
\text{D} & \quad \text{R} & \quad \text{E}
\end{align*}
\]

- Use 2 bars to separate 3 types of stones:
  *|***|*  1 diamond, 3 rubies, 1 emerald
  *****| |  5 diamonds

\[\therefore \text{this problem is same as choosing 2 positions for bars out from 7 possible positions (or choosing 5 positions for * out from 7 positions). i.e.} \quad C(7,5) = C(7,2) \quad C(\text{3*5-1, 5})\]

\[\therefore \text{in general:} \quad n \text{ objects need (}n-1\text{) markers, so} \quad C(r+n-1,r)\]
Example: How many solutions does the equation \( x_1 + x_2 + x_3 = 11 \) have where \( x_i \geq 0 \) integers.

This is same as 11 positions with 2 bars to separate \( x_1, x_2 \) and \( x_3 \). Several examples:

\[
\begin{align*}
1111 & | 1111111 & 11 &= 4 + 5 + 2 \\
11 & | 1 & 11111111 & &= 2 + 1 + 8 \\
11111111111 & | & &= 11 + 0 + 0
\end{align*}
\]

so, \( r = 11 \) and \( n = 3 \)

\[
C(r+n-1, r) = C(r+n-1, n-1) = C(13, 11) = C(13, 2) = 78
\]

**Distinct Permutations from A Set with Repeats**

- Example : How many distinct permutations can be made from the characters in word FLORIDA?
  - all characters are distinct. Answer is \( P(7, 7) = 7! = 5040 \)
  - note: no selection here, use all 7 characters

- Example : How many distinct permutations can be made from the characters in word MISSISSIPPI?
  - It’s not \( 11! \) : \( MIS_1S_2ISSIPPI = MIS_2S_1ISSIPPI, \)
  - need to eliminate duplicates
  - (a) 4 S’s occupy 4 positions in the string, but the arrangement among these 4 S’s does not matter,
    - how many permutations of 4 S’s? \( 4 \times 3 \times 2 \times 1 = 4! \)
  - (b) Similarly, 4 I’s has 4! undistinguished permutations,
  - (c) 2 P’s has 2! undistinguished permutations

Answer: \( 11!/(4! \times 4! \times 2!) = 34650 \)
• Another way to look at it: 

There are 11 positions in a permutation

Choose 4 positions for S: $C(11,4)$ // eleven positions available

Choose 4 positions for I: $C(7,4)$ // only seven positions available

Choose 2 positions for P: $C(3,2)$ // only three positions available

Choose 1 position for M: $C(1,1)$

Note: Can be chosen in any different order, e.g. M, P, I, S

Total ways (product rules): $C(11,4) \times C(7,4) \times C(3,2) \times C(1,1)$

$\Rightarrow \frac{11!}{7! \times 4!} \times \frac{7!}{3! \times 4!} \times \frac{3!}{1! \times 2!} \times \frac{1!}{0! \times 1!}$

$\Rightarrow 11!/(4! \times 4! \times 2! \times 1!)

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Binomial Coefficients

• An application of $C(n, k)$

• A binomial expression is an exponential of the sum of two terms, such as $(a + b)$

$$(a + b)^2 = aa + ab + ba + bb = a^2 + 2ab + b^2$$

$$(a + b)^3 = (a + b)(a + b)(a + b) // 3 positions each with a or b$$

$$(a + b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb$$

$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

There is only one term $a^3$.

Choose $a$ from all three factors: $C(3, 3) = 1$.

There are three times of the term $a^2b$.

Choose $a$ from two out of the three factors: $C(3, 2) = 3$.

Similarly, there are three times of the term $ab^2$.

Choose $a$ from one, $C(3, 1) = 3$, or choose $b$ from two, $C(3, 2) = 3$
This leads us to the following formula,

**Binomial Theorem:**

\[
(a + b)^n = C_0^n a^n b^0 + C_1^n a^{n-1}b^1 + \ldots + C_{r-1}^n a^{n-(r-1)}b^{r-1} + C_r^n a^{n-r}b^r + \ldots + C_n^n a^0 b^n
\]

Note: \( C(n,r) \) is the same as \( \binom{n}{r} \).

**Example:** find the 4th term in \((2x + 5y)^7\).

Let \( a=2x \) and \( b=5y \), we have \(((2x) + (5y))^7\). The 4th term is

\[
\binom{7}{4-1} (2x)^{7-(4-1)} (5y)^{(4-1)}
\]

\[
= \binom{7}{3} (2x)^4 (5y)^3
\]

\[
= \binom{7}{3} 2^4 5^3 x^4 y^3
\]

\[
= 70000 x^4 y^3
\]

**Pascal’s Identity:** Let \( n \) and \( k \) be positive integers with \( n \geq k \). Then \( C(n+1,k) = C(n,k-1) + C(n,k) \).

We can prove by expanding \( C(n,k-1) \) and \( C(n,k) \). Here is the meaning of \( C(n+1,k) \)

\[ T = S \cup \{a\} \]

Imagining a set \( S \) containing \( n \) elements and a set \( T \) containing \( (n + 1) \) elements, namely all elements in \( S \) plus a new element \( a \).

Calculating \( C(n+1,k) \) is equivalent to answering the question: How many subsets of \( T \) containing \( k \) items are there?

Case I: The subset contains \((k - 1)\) elements of \( S \) plus the element \( a \): \( C(n, k-1) \) choices.

Case II: The subset contains \( k \) elements of \( S \) and does not contain \( a \): \( C(n, k) \) choices.

**By Sum Rule:** \( C(n + 1, k) = C(n, k-1) + C(n, k) \).
Row(n)
0  \( C_0^n \)
1  \( C_0^1 \quad C_1^1 \)
2  \( C_0^2 \quad C_1^2 \quad C_2^2 \)
3  \( C_0^3 \quad C_1^3 \quad C_2^3 \quad C_3^3 \)
4  \( C_0^4 \quad C_1^4 \quad C_2^4 \quad C_3^4 \quad C_4^4 \)

\( \vdots \)

\( n \quad C_0^n \quad C_1^n \quad \ldots \quad C_{n-1}^n \quad C_n^n \)

**Row 0:**
1 1

**Row 1:**
1 1 2

**Row 2:**
1 2 1 4

**Row 3:**
1 3 3 1

**Row 4:**
1 4 6 4 1 16

\( C(n+1, k) = C(n, k-1) + C(n, k) \).

Recall Pascal’s Identity:
\( C(3, 0) = C(2, 0) + C(2, 1) \)
\( C(4, 1) = C(3, 0) + C(3, 1) \)

Sum of each row is:
\( C(n, n) = C(n, 0) \)

\( C(n, 0) + C(n, 1) + \ldots + C(n, n) = 2^n \)

**Pascal’s identity with Pascal’s triangle:**

With the help of Pascal’s identity, Pascal triangle can considerably simplify the process of expanding powers of binomial expressions.

For example, the fourth row of Pascal’s triangle \((1 – 4 – 6 – 4 – 1)\) helps us to compute \((a + b)^4\):

\( (a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 \)

i.e. the \(n\)th row of the triangle \((n \geq 0)\) consists of all of the values \( \forall 0 \leq r \leq n \quad C(n, r) \)