Overview

• HW8 due Thursday on 5/10

• Last Lecture
  – Sum Rule: A or B or C
  – Product Rule: A and B and C
  – Principle of Inclusion-Exclusion, Tree Diagram, Pigeon hole principle
  – Permutations
  – Combinations

• This Lecture
  – Combinations continued
  – Permutations and combinations with repeats
  – Pascal’s Identity
  – Binomial Coefficients
  – Pascal Triangle

Summary:

Use \( \binom{n}{r} = \# \text{ of combinations of selecting } r \) distinct objects from \( n \) distinct objects

\[
\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{P(n,r)}{r!} = \binom{n}{r} \cdot \binom{n}{r} = \binom{n}{n-r}.
\]

Note:

\[
\binom{n}{0} = \frac{n!}{(n-0)!} = 1 \quad \binom{n}{1} = \frac{n!}{(n-1)!!} = n \quad \binom{n}{2} = \binom{n}{n-2} \quad \binom{n}{3} = \binom{n}{n-3} \\
\vdots \\
\binom{n}{n} = \frac{n!}{(n-n)!n!} = 1 \quad \binom{n}{r} = \binom{n}{n-r}.
\]
Example: Consider a 5-card hand from a 52-card deck:

How many hands are possible?
Ans: \( \binom{52}{5} = 2,598,960 \)

How many hands consist of all diamonds?
Ans: \( \binom{13}{5} = 1287 \)

How many hands consist of all the same suit (Flush)?
Ans: \( \binom{13}{5} + \binom{13}{5} + \binom{13}{5} + \binom{13}{5} = 4 \times \binom{13}{5} = 5148 \)

How many hands contain three of a kind?
→ exactly 3 of a kind + exactly 4 of a kind
Ans: \( 13 \times \binom{4}{3} \times \binom{48}{2} + 13 \times \binom{4}{4} \times \binom{48}{1} = 58656 + 624 \)

How many hands contain a full house?
→ 3 of a kind with a pair
Ans: \( 13 \times \binom{4}{3} \times 12 \times \binom{4}{2} = 3744 \)

- \( P(n, r) \) and \( \binom{n}{r} \) assume that each object will only be selected once without repetition. **Now, suppose \( n \) objects are available for reuse (or with repetition)**

- **Permutations of \( r \) objects out from \( n \) objects with repetition** is easy:

  \[
  \begin{align*}
  \text{1st object: } & n \text{ choices; } \quad \binom{n}{1} \quad \binom{w}{1} \\
  \text{2nd object: } & n \text{ choices... } \quad \binom{n}{1} \cdot \binom{n-1}{1} \quad \binom{w}{1} \\
  \text{rth object: } & n \text{ choices. } \quad \binom{n}{1} \cdot \binom{n-2}{1} \cdot \ldots \cdot \binom{n-r+1}{1} \quad \binom{w}{1} \\
  \text{Total: } & n^r \quad \frac{\binom{n}{1} \cdot \binom{n-1}{1} \cdot \ldots \cdot \binom{n-r+1}{1}}{n^r} \quad \frac{1}{n!}
  \end{align*}
  \]
Combinations of $r$ objects out from $n$ objects with repetition:

- e.g. a jeweler is designing a pin with 5 stones chosen from diamonds, rubies and emeralds. How many ways can the stones be selected? ($n = 3, r = 5$)
- i.e. 5-combinations with repetition allowed from three-element set.

Use 2 bars to separate 3 types of stones:

```
∗ | ∗∗∗ | ∗
```

1 diamond, 3 rubies, 1 emerald

```
∗∗∗∗∗ | |
```

5 diamonds

∴ this problem is same as choosing 2 positions for bars out from 7 possible positions (or choosing 5 positions for $*$ out from 7 positions). i.e. $C(7,5) = C(7,2)$

∴ in general: $n$ objects need $(n-1)$ markers, so $C(r+n-1, r)$

Example: How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have where $x_i \geq 0$ integers.

This is same as 11 positions with 2 bars to separate $x_1$, $x_2$ and $x_3$. Several examples:

```
1111 | 111111 | 11 = 4 + 5 + 2
11 | 1 | 11111111 = 2 + 1 + 8
11111111111 | | = 11 + 0 + 0
```

so, $r=11$ and $n = 3$

$C(r+n-1, r) = C(r+n-1, n-1) = C(13,11) = C(13,2) = 78$
Distinct Permutations from A Set with Repeats

- Example: How many distinct permutations can be made from the characters in word FLORIDA?
  ∴ all characters are distinct. Answer is \( P(7,7) = 7! = 5040 \)
  note: no selection here, use all 7 characters

- Example: How many distinct permutations can be made from the characters in word MISSISSIPPI?
  It’s not 11! ∴ MISSISSIPPI = M S I S S I P P I,
  ∴ need to eliminate duplicates
  (a) 4 S’s occupy 4 positions in the string, but the arrangement among these 4 S’s does not matter,
  ∴ how many permutations of 4 S’s? 4 * 3 * 2 * 1 = 4!
  (b) Similarly, 4 I’s has 4! undistinguished permutations,
  (c) 2 P’s has 2! undistinguished permutations
  Answer: \( \frac{11!}{4!4!2!} = 34650 \)

Another way to look at it:
There are 11 positions in a permutation
Choose 4 positions for S: \( C(11,4) \) // eleven positions available
Choose 4 positions for I: \( C(7,4) \) // only seven positions available
Choose 2 positions for P: \( C(3,2) \) // only three positions available
Choose 1 position for M: \( C(1,1) \)
Note: Can be chosen in any different order, e.g. M, P, I, S

Total ways (product rules): \( C(11,4) \times C(7,4) \times C(3,2) \times C(1,1) \)

\( \Rightarrow \frac{11!}{(7!4!)} \times \frac{7!}{(3!4!)} \times \frac{3!}{(1!2!)} \times \frac{1!}{(0!1!)} \)

\( \Rightarrow 11!/(4!*4!*2!*1!) \)
Binomial Coefficients

- An application of \( C(n, k) \)
- A binomial expression is an exponential of the sum of two terms, such as \((a + b)\)
  \[
  (a + b)^2 = aa + ab + ba + bb = 1a^2 + 2ab + 1b^2
  \]
  \[
  (a + b)^3 = (a+b)(a+b)(a+b) \quad // \quad 3 \text{ positions each with } a \text{ or } b
  = aaa + aab + aba + abb + baa + bab + bba + bbb
  = 1a^3 + 3a^2b + 3ab^2 + 1b^3
  \]

There is only one term \(a^3\).
  Choose \(a\) from all three factors: \(C(3, 3) = 1\).

There is three times the term \(a^2b\).
  Choose \(a\) from two out of the three factors: \(C(3, 2) = 3\).

Similarly, there is three times the term \(ab^2\)
  Choose \(a\) from one, \(C(3, 1) = 3\), or choose \(b\) from two, \(C(3, 2) = 3\)

This leads us to the following formula,

**Binomial Theorem:**

\[
(a + b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r
\]

Note: \(C(n, r)\) is the same as \(\binom{n}{r}\)

**Example:** find the 4th term in \((2x + 5y)^7\)

Let \(a=2x\) and \(b=5y\), we have \(((2x) + (5y))^7\)
the 4th term is

\[
C_4^7 (2x)^{7-(4-1)} (5y)^{4-1}
\]
\[
= C_3^7 (2x)^4 (5y)^3
\]
\[
= C_3^7 2^4 5^3 x^4 y^3
\]
\[
= 70000 (x^4 y^3)
\]
Pascal’s Identity: Let \( n \) and \( k \) be positive integers with \( n \geq k \). Then
\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.
\]

We can prove by expanding \( \binom{n+1}{k} \) and \( \binom{n}{k} \). Here is the meaning of \( \binom{n+1}{k} \)

Imagine a set \( S \) containing \( n \) elements and a set \( T \) containing \( (n+1) \) elements, namely all elements in \( S \) plus a new element \( a \).

Calculating \( \binom{n+1}{k} \) is equivalent to answering the question: How many subsets of \( T \) containing \( k \) items are there?

Case I: The subset contains \((k-1)\) elements of \( S \) plus the element \( a \): \( \binom{n}{k-1} \) choices.

Case II: The subset contains \( k \) elements of \( S \) and \( \text{not included} \) does not contain \( a \): \( \binom{n}{k} \) choices.

By Sum Rule: \( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \).
• **Pascal’s identity with Pascal’s triangle:**

  With the help of **Pascal’s identity**, Pascal triangle can considerably simplify the process of expanding powers of binomial expressions.

  For example, the fourth row of Pascal’s triangle (1 – 4 – 6 – 4 – 1) helps us to compute \((a + b)^4\):

  \[
  (a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4
  \]

  i.e. the \(n\)th row of the triangle \((n \geq 0)\) consists of all of the values \(\forall 0 \leq r \leq n \ C(n,r)\)