Overview

• HW8 due in a week on 5/14

• Last Lecture
  – Trees

• This Lecture: Counting!
  – Sum Rule: A or B or C
  – Product Rule: A and B and C
  – Principle of Inclusion-Exclusion, Tree Diagram, Pigeonhole principle
  – Permutations
  – Combinations
  – Permutations and combinations with repeats
  – Pascal’s Identity
  – Binomial Coefficients
  – Pascal Triangle

Chapter 7. Counting: Combinatorics
• **Combinatorics** is a branch of mathematics that deals with counting. Counting is an important basic part of Discrete Math:

Examples:

“How many possible ways are there to pick 11 soccer players out of a 20-player team?”

“How many different 8-letter passwords are there?”

Most importantly, counting is the basis for computing **probabilities** of discrete events.

(“What is the probability of winning the lottery?”)

---

**Basic Counting Principles**

• The sum rule: If a task $T_1$ can be done in $n_1$ ways and a second task $T_2$ in $n_2$ ways, and if these two tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task.

Example:

The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors? There are $530 + 15 = 545$ choices.

• Generalized sum rule: If we have tasks $T_1, T_2, ..., T_m$ that can be done in $n_1, n_2, ..., n_m$ ways, respectively, and no two of these tasks can be done at the same time, then there are $n_1 + n_2 + ... + n_m$ ways to do one of these tasks.

$$n_1n_2n_3...n_m = \sum_{i=1}^{m} n_i$$
• **The product rule:** Suppose that a procedure can be broken down into two successive tasks $T_1$ and $T_2$. If there are $n_1$ ways to do the first task and $n_2$ ways to do the second task after the first task has been done, then there are $n_1n_2$ ways to do the procedure.

$$T_1 \text{ and } T_2 \rightarrow n_1 \times n_2$$

• **Example:** How many different license plates are there that containing exactly two English letters?

• **Solution:** There are 26 possibilities to pick the first letter, then 26 possibilities for the second one. So there are $26 \times 26 = 676$ different license plates.

• **Generalized product rule:** If we have a procedure consisting of sequential tasks $T_1, T_2, ..., T_m$ that can be done in $n_1, n_2, ..., n_m$ ways, respectively, then there are $n_1 \times n_2 \times ... \times n_m$ ways to carry out the procedure.

$$\prod_{i=1}^{m} n_i$$

**Example:** The last part of your phone # contains 4 digits. How many four-digit numbers are there (ATM PIN#)? (Select a password of 4 digits, how many possible passwords?)

Some 4 digit numbers: 9933, 1982, 2017, 1016, 0007, ...

$10 \times 10 \times 10 \times 10 = 10,000$ different numbers outcomes

**Example:** 10 toppings available in a pizza store, and the customer can choose any # of toppings, how many different pizza the store can make?

<table>
<thead>
<tr>
<th>Toppings</th>
<th>choose it</th>
<th>or w/o it</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Choose it $2^{10} = 1,024$

Or w/o it $2^{10} = 1,024$
• The sum and product rules can also be phrased in terms of set theory.

• **Sum rule**: Let $A_1, A_2, \ldots, A_m$ be disjoint sets. Then the number of ways to choose any element from one of these sets is
  \[|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|\]

• **Product rule**: Let $A_1, A_2, \ldots, A_m$ be finite sets. Then the number of ways to choose one element from each set in the order $A_1, A_2, \ldots, A_m$ is
  \[|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \times |A_2| \times \ldots \times |A_m|\]

• Next, we look at various methods to compute results using combination of sum rule and product rule.

---

**Example: How many ways to choose two courses from CSC212, CSC210, CSC211. If 6 sections of CSC212, 7 sections of CSC210 and 5 sections of CSC211 are offered.**

- # of ways to select CSC212 & CSC210 : 6*7 = 42
- # of ways to select CSC212 & CSC211 : 6*5 = 30
- # of ways to select CSC210 & CSC211 : 7*5 = 35
- Total: 107 ways

**Example: How many 3-letter words have a letter repeated if words are formed from \{a,b,c,d,e\}.**

- a. Total # of 3-letter words : 5*5*5=125
- b. Total # of 3-letter words with distinct letters : 5*4*3=60
- c. Total # of words with repeated letters: (a)-(b) = 65

---

**Note: counting the complement**

- No repeats abc, edb, \ldots
- One repeat: jeb, daa, \ldots
- All repeat: aaa,edd, \ldots
- Good trick to save time!

**Selected 2 letters: 20+5*4**

For each 2 letters you have 3 ways to order a,b, a,b, b,a

**ANS**: 65
Tree Diagrams
• How many bit strings of length four do not have two consecutive 1s?

  1st bit  2nd bit  3rd bit  4th bit
  0      0      0      0, 1
  1      0      1      0, 1
  1      1      0      0
  1      0      1      0
  0      1      0      0
  0      0      1      0
  0      0      0      1

There are 8 strings.

Recall Principle of Inclusion-Exclusion
• Recall this in set theory: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
• How many bit strings of length 8 either start with a 1 or end with 00?

  Task 1: Construct a string of length 8 that starts with a 1.
  $1*2*2*2*2*2*2*2$
  Product rule: Task 1 can be done in $1\cdot2^7 = 128$ ways.

  Task 2: Construct a string of length 8 that ends with 00.
  $2*2*2*2*2*1*1$
  Product rule: Task 2 can be done in $2^5 = 64$ ways.
Task 3: How many strings start with 1 and end with 00?

There is one way to pick the first bit (1), two ways for the second, ..., sixth bit (0 or 1), one way for the seventh, eighth bit (0).

$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1 \times 1$

Product rule: In $2^5 = 32$ cases

Total number of strings is $128 + 64 - 32 = 160$

More examples:
- How many bit strings of length 8? $2^8 = 256$
- How many bit strings of length 8 with no 1? 1
- How many bit strings of length 8 with exactly one 1? 8
- How many bit strings of length 8 with exactly two 1s?
  - 1_ _ _ _ _ _ _ // 1st position and 7 other positions
  - X 1 _ _ _ _ _ // 2nd position and 6 other positions
  - ... By Sum Rule
  - $1 + 8 + 28 = 37$
- How many bit strings of length 8 with at most two 1s?
  - $1 + 8 + 28 = 37$
- How many bit strings of length 8 with at least three 1s?
  - $256 - 37 = 219$
Recall The Pigeonhole Principle

Example: A bank requires all customers to choose a four-digit code to use with an ATM card. The code must contain two letters in the first two positions and two numbers in the last two positions. The bank has 75,000 customers. Show that at least two customers choose the same four-digit code.

Total number of codes: $26 \times 26 \times 10 \times 10 = 67,600$. By the Pigeonhole Principle, there are at least two customers who choose the same code.

Permutations

• A permutation is an ordered arrangement of objects (without repetition).
  
• An ordered arrangement of $r$ elements ($r < |S|$) of a set $S$ is called an $r$-permutation. For example, (3, 2) or (2, 1, 3) are two different examples of permutation. (3, 2) or (1, 3) or (2, 3) are three examples of 2-permutation. (1) or (2) or (3) are examples of 1-permutation.
• The number of r-permutations of a set with n distinct elements is denoted by \( P(n, r) \) ("n P r"). We can calculate \( P(n, r) \) with the product rule:

\[
P(n, r) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)
\]

\( (n \text{ choices for the first element, } (n-1) \text{ for the second one, } (n-2) \text{ for the third one...}) \)

• General formula: \( P(n, r) = \frac{n!}{(n-r)!} \)

Note: \( P(n, 0) = \frac{n!}{(n-0)!} = 1 \)

\[ P(n, 1) = \frac{n!}{(n-1)!} = n \]

\[ P(n, n) = \frac{n!}{(n-n)!} = n! \cdot 0! = 1 \]

• Example:

\[ P(8, 3) = 8 \cdot 7 \cdot 6 = 336 = \frac{8! \cdot 6!}{5!} = 8 \cdot 7 \cdot 6 = 336 \]

• Example: How many 3-letter words (not necessarily meaningful) can be formed from the word "Compiler" if no letter can be repeated?

\[ P(8, 3) = \frac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336 \]

• Example: How many ways can a president and vice-president be selected from a group of 20 people?

\[ P(20, 2) = \frac{20!}{18!} = 20 \cdot 19 = 380 \]
Overview

• HW8 due on 5/14

• Last Lecture
  – Sum Rule: A or B or C
  – Product Rule: A and B and C
  – Principle of Inclusion-Exclusion, Tree Diagram, Pigeonhole principle
  – Permutations
    \[ P(n, r) = \frac{n!}{(n-r)!} \]

• This Lecture: Counting!
  – Combination
  – Permutations and combinations with repeats
  – Pascal’s Identity
  – Binomial Coefficients
  – Pascal Triangle

Combinations

• Example: How many ways are there to pick a set of 3 people from a group of 6?

There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so there are \(6 \times 5 \times 4 = 120\) ways to do this.

This is not the correct result! \( \binom{6}{3} = \frac{6!}{3! \times 3!} = 20 \)

For example, picking person C, then person A, and then person E leads to the same group as first picking E, then C, and then A.

However, these cases are counted separately in the above formulation.

So how can we compute how many different subsets of people can be picked (i.e., we want to disregard the order of picking)?
An \( r \)-combination of elements of a set is an **unordered** selection of \( r \) elements from the set. Thus, an \( r \)-combination is simply a **subset** of the set with \( r \) elements.

Example: Let \( S = \{1, 2, 3, 4\} \). Then \( \{1, 3, 4\} = \{3, 1, 4\} \) is an example member of 3-combination.

The **number of all possible** \( r \)-combinations from a set with \( n \) distinct elements is denoted by \( \binom{n}{r} \) ("\( n \) choose \( r \)).

Example: \( \binom{4}{2} = 6 \), i.e. the 2-combinations of a set \( \{1, 2, 3, 4\} \) are \( \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \).

Now we can answer our question: How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

\[ \binom{6}{3} = \frac{6!}{3! \cdot 3!} = \frac{6 	imes 5 	imes 4}{3 	imes 2 	imes 1} = 20 \]

Corollary: Let \( n \) and \( r \) be nonnegative integers with \( r \leq n \). Then \( \binom{n}{r} = \binom{n}{n-r} \).

Note: each choice \( r \)-elements determine \( (n-r) \)-choose choice of \( (n-r) \)-elements
Summary:

Use \( C(n, r) = \# \text{ of combinations of selecting } r \)

\( \text{distinct objects from } n \text{ distinct objects} \)

\[
C(n, r) = \frac{n!}{(n-r)!r!} = \frac{P(n, r)}{r!} = C(n, r)
\]

Note:

\( C(n, 0) = \frac{n!}{(n-0)!} = 1 \)

\( C(n, 1) = \frac{n!}{(n-1)!} = n \)

\( C(n, n) = \frac{n!}{(n-n)!n!} = 1 \)

\( C(n, r) = C(n, n-r). \)

• Example: A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

\[ \text{Ans: } C(8, 6) * C(7, 5) = \frac{8!}{6!2!} * \frac{7!}{5!2!} = 28 * 21 = 588 \]

• Example: A committee of 8 students is to be selected from a class consisting of 19 freshmen and 34 sophomores. In how many ways can 3 freshmen and 5 sophomores be selected? \( \text{Ans: } C(19,3) * C(34,5) \)

In how many ways can a committee with exactly 1 freshman be selected? \( \text{Ans: } C(19,1) * C(34,7) \)

In how many ways can a committee with at most 1 freshman be selected? \( 0 \text{ freshman + 1 freshman } = C(34,8) + C(19,1) * C(34,7) \)

In how many ways can a committee with at least 1 freshman be selected? All 8-combination – no freshman = \( C(53,8) - C(34,8) \)
Recall from previous examples:

How many bit strings of length 8 with exactly one 1?
This problem is same as:
how many 1-combination from set \{1,2,3,4,5,6,7,8\}
where each element is a position in bit strings?
Answer: C(8,1) = 8

How many bit strings of length 8 with exactly two 1s?
Answer: C(8,2) = \(7 + 6 + 5 + 4 + 3 + 2 + 1\)

How many bit strings of length 8 with exactly r 1s?
Answer: C(8,r)

How many bit strings of length 8 with at most two 1s?
Answer: C(8,0)+C(8,1)+C(8,2)

Example: Consider a 5-card hand from a 52-card deck:
How many hands are possible?
Ans: C(52,5) = 2,598,960

How many hands consist of all diamonds?
Ans: C(13,5) = 1287

How many hands consist of all the same suit (Flush)?
Ans: C(13,5) + C(13,5) + C(13,5) + C(13,5) = 5148

How many hands contain three of a kind?
→ exactly 3 of a kind + exactly 4 of a kind
Ans: 13*C(4,3)*C(48,2) + 13*C(4,4)*C(48,1) = 58656 + 624

How many hands contain a full house?
→ 3 of a kind with a pair
Ans: 13*C(4,3)*12*C(4,2) = 3744
• \( P(n, r) \) and \( C(n, r) \) assume that each object will only be selected once without repetition. **Now, suppose \( n \) objects are available for reuse (or with repetition)**

• **Permutations of \( r \) objects out from \( n \) objects with repetition** is easy:

  1\textsuperscript{st} object: \( n \) choices; 
  2\textsuperscript{nd} object: \( n \) choices... 
  \( r \textsuperscript{th} \) object: \( n \) choices.

  Total: \( n^r \)

\[
\begin{align*}
\text{1st object: } & n \text{ choices} \\
\text{2nd object: } & n \text{ choices} \\
\vdots \\
\text{rth object: } & n \text{ choices} \\
\text{Total: } & n^r
\end{align*}
\]

\[
\begin{align*}
\frac{n!}{r!(n-r)!} = \binom{n}{r} \quad (n \geq r, \text{def})
\end{align*}
\]

**Combinations of \( r \) objects out from \( n \) objects with repetition:**

e.g. a jeweler is designing a pin with 5 stones chosen from diamonds, rubies and emeralds. How many ways can the stones be selected? \( (n = 3, r = 5) \)

i.e. 5-combinations with repetition allowed from three-element set.

\[
\begin{align*}
\begin{array}{c}
\text{D} \\
\text{R} \\
\text{E}
\end{array}
\end{align*}
\]

use 2 bars to separate 3 types of stones:

\[
\begin{align*}
\text{*|***|*} & \quad 1 \text{ diamond, 3 rubies, 1 emeralds} \\
\text{*****| |} & \quad 5 \text{ diamonds}
\end{align*}
\]

\[
\therefore \text{this problem is same as choosing } 2 \text{ positions for bars out from } 7 \text{ possible positions (or choosing } 5 \text{ positions for * out from } 7 \text{ positions). i.e. } C(7,5) = C(7,2)
\]

\[
\therefore \text{in general: } n \text{ objects need } (n-1) \text{ markers, so } C(r+n-1, r)
\]
Example: How many solutions does the equation \( x_1 + x_2 + x_3 = 11 \) have where \( x_i \geq 0 \) integers.

This is same as 11 positions with 2 bars to separate \( x_1 \), \( x_2 \) and \( x_3 \). Several examples:

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
</tr>
</tbody>
</table>

so, \( r=11 \) and \( n=3 \)

\[ C(r+n-1, r) = C(r+n-1, n-1) = C(13,11) = C(13,2) = 78 \]

**Distinct Permutations from A Set with Repeats**

- Example: How many distinct permutations can be made from the characters in word FLORIDA?
  \( \because \) all characters are distinct. Answer is \( P(7,7) = 7! = 5040 \)
  note: no selection here, use all 7 characters

- Example: How many distinct permutations can be made from the characters in word MISSISSIPPI?
  It’s not 11! \( \because \) MISSISSIPPI = MISISSISSIPPI,
  \( \therefore \) need to eliminate duplicates
  (a) 4 \( S \)'s occupy 4 positions in the string, but the arrangement among these 4 \( S \)'s does not matter,
  \( \therefore \) how many permutations of 4 \( S \)'s? \( 4! \times 3! = 4! \)
  (b) Similarly, 4 \( I \)'s has 4! undistinguished permutations,
  (c) 2\( P \)'s has 2! undistinguished permutations
  Answer: \( 11! / (4! \times 4! \times 2!) = 34650 \)
Another way to look at it: 

There are 11 positions in a permutation

Choose 4 positions for S: \( C(11,4) \) // eleven positions available

Choose 4 positions for I: \( C(7,4) \) // only seven positions available

Choose 2 positions for P: \( C(3,2) \) // only three positions available

Choose 1 position for M: \( C(1,1) \)

Note: Can be chosen in any different order, e.g. M, P, I, S

Total ways (product rules): \( C(11,4) \times C(7,4) \times C(3,2) \times C(1,1) \)

\[ \frac{11!}{4!4!2!1!} \]

Binomial Coefficients

- An application of \( C(n, k) \)

- A binomial expression is an exponential of the sum of two terms, such as

\[
(a + b)^2 = ab + ba + bb = 1a^2 + 2ab + 1b^2
\]

\[
(a + b)^3 = (a+b)(a+b)(a+b) // 3 positions each with a or b
= a + a + a + a + b + b + b + b + b + b + b + b
= 1a^3 + 3a^2b + 3ab^2 + 1b^3
\]

There is only one term \( a^3 \).

Choose \( a \) from all three factors: \( C(3,3) = 1 \).

There are three times of the term \( a^2b \).

Choose \( a \) from two out of the three factors: \( C(3,2) = 3 \).

Similarly, there are three times of the term \( ab^2 \)

Choose \( a \) from one, \( C(3,1) = 3 \), or choose \( b \) from two, \( C(3,2) = 3 \)
This leads us to the following formula,

**Binomial Theorem:**

\[
(a + b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}a^0 b^n
\]

**Example:** Find the 4th term in \((2x + 5y)^7\).

Let \(a = 2x\) and \(b = 5y\), we have \(((2x) + (5y))^7\). The 4th term is

\[
\binom{7}{4-1} (2x)^7 (4-1) (5y)^{4-1} = \binom{7}{3} (2x)^4 (5y)^3 = \binom{7}{3} 2^4 5^3 x^4 y^3 = 70000 x^4 y^3
\]

**Pascal's Identity:** Let \(n\) and \(k\) be positive integers with \(n \geq k\). Then \(C(n+1, k) = C(n, k-1) + C(n, k)\).

We can prove by expanding \(C(n,k-1)\) and \(C(n,k)\). Here is the meaning of \(C(n+1,k)\)

\[
C(6,2,27) = C(6,2,25) + C(6,2,27)
\]

Imagine a set \(S\) containing \(n\) elements and a set \(T\) containing \((n + 1)\) elements, namely all elements in \(S\) plus a new element \(a\).

Calculating \(C(n + 1, k)\) is equivalent to answering the question: How many subsets of \(T\) containing \(k\) items are there?

**Case I:** The subset contains \((k - 1)\) elements of \(S\) plus the element \(a\): \(C(n, k - 1)\) choices.

**Case II:** The subset contains \(k\) elements of \(S\) and does not contain \(a\): \(C(n, k)\) choices.

**By Sum Rule:** \(C(n + 1, k) = C(n, k - 1) + C(n, k)\).
Recall Pascal’s identity:
\[ \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \]

Support Pascal’s identity with Pascal’s triangle:

With the help of Pascal’s identity, Pascal triangle can considerably simplify the process of expanding powers of binomial expressions.

For example, the fourth row of Pascal’s triangle 
\((1 – 4 – 6 – 4 – 1)\) helps us to compute \((a + b)^4\):

\[
(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4
\]

i.e. the \(n\)th row of the triangle \((n\geq0)\) consists of all of the values \(\forall 0 \leq r \leq n \; \binom{n}{r}\)