Overview

• HW7 Due & HW8 assignment will be online!
• HW8 is due in one week on 12/8 Tuesday

• Last Lecture
  – Trees

• This Lecture: Counting!
  – Sum Rule: $A$ or $B$ or $C$
  – Product Rule: $A$ and $B$ and $C$
  – Principle of Inclusion-Exclusion, Tree Diagram, Pigeonhole principle
  – Permutations
  – Combinations

Chapter 7. Counting: Combinatorics
**Combinatorics** is a branch of mathematics that deals with counting. Counting is an important basic part of Discrete Math:

Examples:

“How many possible ways are there to pick 11 soccer players out of a 20-player team?”

“How many different 8-letter passwords are there?”

Most importantly, counting is the basis for computing **probabilities** of discrete events.

(“What is the probability of winning the lottery?”)

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**Basic Counting Principles**

- **The sum rule**: If a task \( T_1 \) can be done in \( n_1 \) ways and a second task \( T_2 \) in \( n_2 \) ways, **and if these two tasks cannot be done at the same time**, then there are \( n_1 + n_2 \) ways to do either task.

- **Example**: The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors? There are \( 530 + 15 = 545 \) choices.

- **Generalized sum rule**: If we have tasks \( T_1, T_2, ..., T_m \) that can be done in \( n_1, n_2, ..., n_m \) ways, respectively, and no two of these tasks can be done at the same time, then there are \( n_1 + n_2 + ... + n_m \) ways to do one of these tasks.
• **The product rule:** Suppose that a procedure can be broken down into two successive tasks $T_1$ and $T_2$. If there are $n_1$ ways to do the first task and $n_2$ ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

- **Example:** How many different license plates are there that containing exactly two English letters?

- **Solution:** There are 26 possibilities to pick the first letter, then 26 possibilities for the second one. So there are $26 \cdot 26 = 676$ different license plates.

• **Generalized product rule:** If we have a procedure consisting of sequential tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to carry out the procedure.

- **Example:** The last part of your phone # contains 4 digits. How many four-digit numbers are there (ATM PIN#)? (Select a password of 4 digits, how many possible passwords?)

  some 4 digit numbers: 9933, 1982, 2017, 1016, 0007, …

  \[ 10 \times 10 \times 10 \times 10 = 10,000 \text{ different numbers outcomes} \]

- **Example:** 10 toppings available in a pizza store, and the customer can choose any # of toppings, how many different pizza the store can make?

  Toppings | pepper | tomato | cheese | … | mushroom
 ---------|-------|--------|--------|---|---------
  choose it | 2 * 2 * 2 * … * 2 = $2^{10}$
  or w/o it | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | … | $\frac{1}{2}$
  = 1,024
• The sum and product rules can also be phrased in terms of set theory.

• **Sum rule:** Let \( A_1, A_2, \ldots, A_m \) be disjoint sets. Then the number of ways to choose any element from one of these sets is

\[
|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|
\]

• **Product rule:** Let \( A_1, A_2, \ldots, A_m \) be finite sets. Then the number of ways to choose one element from each set in the order \( A_1, A_2, \ldots, A_m \) is

\[
|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \times |A_2| \times \ldots \times |A_m|
\]

• Next, we look at various methods to compute results using combination of sum rule and product rule

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**Example:** How many ways to choose two courses from CSC212, CSC210, CSC211. If 6 sections of CSC212, 7 sections of CSC210 and 5 sections of CSC211 are offered.

- # of ways to select CSC212 & CSC210 : \(6 \times 7 = 42\)
- # of ways to select CSC212 & CSC211 : \(6 \times 5 = 30\)
- # of ways to select CSC210 & CSC211 : \(7 \times 5 = 35\)
- total: 107 ways

**Example:** How many 3-letter words have a letter repeated if words are formed from \{a, b, c, d, e\}.

a. Total # of 3-letter words : \(5 \times 5 \times 5 = 125\)

b. Total # of 3-letter words with distinct letters : \(5 \times 4 \times 3 = 60\)

c. Total # of words with repeated letters: (a)-(b) = 65

**Note: counting the complement**
Tree Diagrams

- How many bit strings of length four do not have two consecutive 1s?

Recall Principle of Inclusion-Exclusion

- Recall this in set theory: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

- How many bit strings of length 8 either start with a 1 or end with 00?

**Task 1:** Construct a string of length 8 that starts with a 1.
$1*2*2*2*2*2*2*2$
**Product rule:** Task 1 can be done in $1^5 \times 2^3 = 128$ ways.

**Task 2:** Construct a string of length 8 that ends with 00.
$2*2*2*2*2*2*1*1$
**Product rule:** Task 2 can be done in $2^5 = 64$ ways.
Task 3: How many strings start with 1 and end with 00?

There is one way to pick the first bit (1),
two ways for the second, ..., sixth bit (0 or 1),
one way for the seventh, eighth bit (0).
\[1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1 \times 1\]

Product rule: \(2^5 = 32\) cases

Total number of strings is \(128 + 64 - 32 = 160\)

More examples:

How many bit strings of length 8? \(2^8 = 256\)

How many bit strings of length 8 with no 1? 1

How many bit strings of length 8 with exactly one 1? 8

How many bit strings of length 8 with exactly two 1s?

- By Sun Rule
  \[7 + 6 + 5 + 4 + 3 + 2 + 1 = 28\] or \(C(8, 2)\) // will cover \(C()\) later

How many bit strings of length 8 with at most two 1s?

- By Sun Rule
  \[1 + 8 + 28 = 37\]

How many bit strings of length 8 with at least three 1s?

- By Sun Rule
  \[256 - 37 = 219\]
Recall The Pigeonhole Principle

Example: A bank requires all customers to choose a four-digit code to use with an ATM card. The code must contain two letters in first two positions and two numbers in last two positions. The bank has 75000 customers. Show that at least two customers choose the same four-digit code.

Total number of codes: \(26 \times 26 \times 10 \times 10 = 67600\).

By the Pigeonhole Principle, there are at least two customers who choose the same code.

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Permutations

- A **permutation** is an ordered arrangement of objects. (without repetition)

- An ordered arrangement of \(r\) elements \((r < |S|)\) of a set \(S\) is called an \(r\)-permutation. \((3,1,2)\) or \((2,1,3)\) are two different examples of permutation. \((3,2)\) or \((1,3)\) or \((2,3)\) are three examples of 2-permutation. 

- Example: Let \(S = \{1, 2, 3\}\). \((1, 2, 3), (2, 3, 1)\) are two different examples of permutation. \((3, 2)\) or \((1, 3)\) or \((2, 3)\) or \((3, 1)\) or \((2, 3)\) or \((1, 3)\) is a 3-permutation.
• The number of r-permutations of a set with n distinct elements is denoted by \( P(n, r) \) ("\( nP_r \)). We can calculate \( P(n, r) \) with the product rule: \( P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - r + 1) \).

\( n \) choices for the first element, \( (n - 1) \) for the second one, \( (n - 2) \) for the third one...

\[ P(n, r) = \frac{n!}{(n-r)!} \]

Note:  
\[ P(n, 0) = \frac{n!}{(n-0)!} = 1 \]
\[ P(n, 1) = \frac{n!}{(n-1)!} = n \]
\[ P(n, n) = \frac{n!}{(n-n)!} = n! : 0! = 1 \]

• Example:

\[ P(8, 3) = 8 \cdot 7 \cdot 6 = 336 = (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \]

• Example: How many 3-letter words (not necessarily meaningful) can be formed from the word “Compiler” if no letter can be repeated?

\[ P(8, 3) = \frac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336 \]

• Example: How many ways can a president and vice-president be selected from a group of 20 people?

\[ P(20, 2) = \frac{20!}{18!} = 20 \cdot 19 = 380 \]
Combinations

• Example: How many ways are there to pick a set of 3 people from a group of 6?

There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so there are \(6 \cdot 5 \cdot 4 = 120\) ways to do this.

This is not the correct result! \(\binom{6}{2,3} \neq \binom{6}{2,3}\)

For example, picking person C, then person A, and then person E leads to the same group as first picking E, then C, and then A.

However, these cases are counted separately in the above formulation.

So how can we compute how many different subsets of people can be picked (i.e., we want to disregard the order of picking)?
• How can we derive \( C(n, r) \)? Formula?

Consider that we can obtain \( P(n, r) \) of a set in the following way:

First, find all \( r \)-combinations of the set, i.e. \( C(n, r) \)

Then, for each \( r \)-combination, generate all possible orderings, i.e. \( P(r, r) = r! \). Therefore, we have:

\[
C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)! \cdot r!} = \frac{n!}{r!(n-r)!} \cdot \frac{P(n, r)}{r!} \]

• Now we can answer our question: How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

\[
P(6, 3) = \frac{6!}{(6-3)! \cdot 3!} = \frac{720}{6 \cdot 6} = 20
\]

• Corollary: Let \( n \) and \( r \) be nonnegative integers with \( r \leq n \). Then \( C(n, r) = C(n, n-r) \).

Note: each choice \( r \)-elements determine a unique choice of \((n-r)\)-elements.

Summary:

Use \( C(n, r) \) = # of combinations of selecting \( r \) distinct objects from \( n \) distinct objects:

\[
C(n, r) = \frac{n!}{(n-r)! \cdot r!}
\]

Note:

\[
C(n, 0) = \frac{n!}{(n-0)!} = 1
\]

\[
C(n, 1) = \frac{n!}{(n-1)!} = n
\]

\[
C(n, n) = \frac{n!}{(n-n)! \cdot n!} = 1
\]

\[
C(n, r) = C(n, n-r).
\]