Midterm #3

- HW7 Due Now & HW8 questions online soon: Due on 5/14!
- Midterm #3 on Tuesday 5/5 (Algorithm, Numbers, Graphs, Trees)
  - On ilearn/6 questions on Algorithms, Numbers, Graphs and Trees.
  - Open hand-written notes, calculator, plenty of scratch papers
  - No leaving Zoom view: please go to bathroom beforehand!
  - Submission Problems: Make sure you double check all pages are readable and present by downloading and checking what you submitted before you leave.
  - Study with example problems in Lec Notes, HWs, TextBook
  - Study beyond HW questions, studying the examples from lec. notes.
- This Lecture
  - Review for Midterm #3
  - HW6 & HW7 Answers

Homework 6
(Total 25 pts)

CSC230 Discrete Math
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HW#6 Q2 (i) (3pt)  \( f(x) = O(g(x)) \)

Show that (i) \( x^3 \) is \( O(x^4) \)

• **ANS**

\[
\frac{x^3}{x^4} \leq C \quad \text{for all } x \geq 1
\]

It is true that

\[
x^3 \leq x^4 \quad \text{for all } x \geq 1
\]

This demonstrates that \( x^3 \leq x^4 \) is true when \( C=1 \) & \( d=1 \).

\( O(x^4) \)

\[
1 < 10^{-9} < 10^{-8} < 10^{-7} < 10^{-6} < 10^{-5} < 10^{-4} < 10^{-3} < 10^{-2} < 10^{-1} < 1 < n
\]

HW#6 Q2 (ii) (3pt)  \( f(x) \neq O(g(x)) \)

Show that (ii) \( x^4 \) is not \( O(x^3) \)

• **ANS**

\[
\frac{x^4}{x^3} \neq C \quad \text{for any constant } C \text{ for all } x \geq 1
\]

It is straightforward to show that

\[
x^4 \neq C \quad \text{for any } C \geq 3 \text{ for all } x \geq 1
\]

Multiply both sides of the inequality \( x^4 \geq d \geq 1 \). Thus \( x^4 \) is not \( O(x^3) \)
HW#6 Q4 (7pt)
Analyze running time of binary search (the algorithm is given in class notes). You must explain your answers.

• ANS
Binary search algorithm searches through a list of \( n \) integers. In each loop of the algorithm, the size of list is reduced by \( \frac{1}{2} \) until it becomes one. Thus

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
\text{iteration} & 1 & 2 & 3 & \ldots & \left\lfloor \frac{n}{2} \right\rfloor & \ldots & \left\lfloor \frac{n}{2^k} \right\rfloor & 1 \\
\text{list size} & n & \frac{n}{2} & \frac{n}{2^2} & \ldots & \frac{n}{2^k} & \ldots & \frac{n}{2^k} & 1 \\
\end{array}
\]
Suppose the max # of iterations was \( k \). Then we have

\[
2^k \leq n \iff 2^k = n \iff \log_2 2^k = \log_2 n \iff k = \log_2 n \iff b = \log_2 n
\]
So the complexity function of Binary search is \( \log_2 n + 1 \)
Since \( \log_2 n + 1 \leq c \log_2 n \) when \( c = 2, n \geq 2 \), it is \( O(\log n) \)

HW#6 Q6(a) (3pt)
Find a counter example to a statement:
If \( ac \equiv bc \pmod{m} \), \( a,b,c,m \) are integers, \( m \geq 2 \),
then \( a \equiv b \pmod{m} \)

• ANS
\( a = 0, b = 1, c = 2, m = 2 \)
\[ a \equiv b \pmod{m} \iff m \mid a-b \iff 2 \mid a-b \]
\[ ac \equiv bc \pmod{m} \iff m \mid ac-bc \iff 2 \mid (a-b) \]
\[ \therefore a \equiv b \pmod{m} \iff m \mid a-b \iff 2f-1 \]
HW#6 Q6 (b) (3pt)

Find a counter example to a statement:
If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, $a,b,c,m$ are integers, $c,d>0$, $m \geq 2$, then $a^c \equiv b^d \pmod{m}$

• ANS

$\begin{align*}
    a &= 3, \quad b = 3, \quad c = 1, \quad d = 6, \quad m = 5 \\
    \therefore a^2 b \equiv (a \cdot d) &\iff \text{ gcd } (a-b) \quad \rightarrow \quad 5 \mid 0 \\
    c^2 d \equiv (c \cdot d) &\iff \text{ gcd } (c-d) \quad \rightarrow \quad 5 \mid 5 \\
    a^c \equiv b^d \pmod{m} &\iff \text{ gcd } (a^c - b^d) \quad \rightarrow \quad 5 \mid (3^1 - 3^6) \\
    \Leftrightarrow &\quad 5 \mid 726
\end{align*}$

HW#6 Q7 (6pt)

Find gcd(1000,625) and lcm(1000,625) and verify that $\text{gcd}(1000,625) \cdot \text{lcm}(1000,625) = 625 \times 10^3$

• ANS

$\begin{align*}
    \text{gcd}(1000,625) &= 625 \\
    \text{lcm}(1000,625) &= 5000 & \left( \text{you can use Euclidean Alg} \right)
\end{align*}$

Thus

$\begin{align*}
    \text{gcd}(1000,625) \cdot \text{lcm}(1000,625) &= 5000 \\
    &= 625 \times 5000 \\
    &= 625000
\end{align*}$
HW#7 Q1 (3pt)

Show that the sum, over the set of people at a party, of the number of people a person has shaken hands with, is even. Assume that no one shake his or her own hands.

- **ANS**

Represent this problem by a graph whose vertex is a person and an edge between two people represents that they shake hands. Then the degree of a vertex is equivalent to the number of people that the person (vertex) shakes hand with.

By the handshaking theorem, the sum of the degrees of all vertices is even:

\[ 2e = \sum_{v \in V} \deg(v) \]

Even # \( \iff \) multiple of 2. Question's sum
HW#7 Q2-a (2pt)

Determine this pair of graph is isomorphic or not.

- ANS

HW#7 Q2-b (2pt)

Determine this pair of graph isomorphic or not.

- ANS
HW#7 Q2-c (2pt)

Determine this pair of graph isomorphic or not.

- ANS

\[ |V| = 6 \]
\[ |E| = 9 \]

\[ |V_1| = 3 \]
\[ |E_1| = 3 \]

A and B are isomorphic. An example of bijection is:

1. \( f(u_1) = v_5 \)
2. \( f(u_2) = v_2 \)
3. \( f(u_3) = v_3 \)
4. \( f(u_4) = v_6 \)
5. \( f(u_5) = v_4 \)
6. \( f(u_6) = v_1 \)

HW#7 Q3 (6pt)

Find a shortest path between 'a' and 'z' in the following graph using Dijkstra's algorithm.

- ANS

1. \( d[a] = 0 \)
2. \( d[b] = 1 \)
3. \( d[c] = 2 \)
4. \( d[d] = 3 \)
5. \( d[e] = 4 \)
6. \( d[f] = 5 \)
7. \( d[g] = 6 \)
8. \( d[h] = 7 \)
9. \( d[i] = 8 \)
10. \( d[j] = 9 \)
11. \( d[k] = 10 \)
12. \( d[l] = 11 \)
13. \( d[m] = 12 \)
14. \( d[n] = 13 \)
15. \( d[o] = 14 \)
16. \( d[p] = 15 \)
17. \( d[q] = 16 \)
18. \( d[r] = 17 \)
19. \( d[s] = 18 \)
20. \( d[t] = 19 \)
21. \( d[u] = 20 \)
22. \( d[v] = 21 \)
23. \( d[w] = 22 \)
24. \( d[x] = 23 \)
25. \( d[y] = 24 \)
26. \( d[z] = 25 \)

The shortest path: \( a \to c \to d \to e \to g \to z \). Path cost: 12.
HW#7 Q4-a (2pt)

Determine whether the given graph is planar

- ANS

\[ P_{C^3} \]
\[ d(C) \]
HW#7 Q4-b (2pt)

Determine whether the given graph is planar

- ANS

\[ \text{YES} \]

\[ \begin{array}{c}
  a & b & c \\
  d & e & f \\
\end{array} \]

\[ \Rightarrow \]

\[ \begin{array}{c}
  a & d & b \\
  c & e & f \\
\end{array} \]

HW#7 Q5-a (2pt)

List vertices using inorder traversal

- ANS

\[ \text{jeknopbfacdglm} \]
HW#7 Q5-b (2pt)

List vertices using preorder traversal
• ANS

HW#7 Q5-c (2pt)

List vertices using postorder traversal
• ANS