Midterm #2

- HW6: Due on 11/7 after MT2.
- Midterm #2 in Five Days
  (Relation/Function/CountingTheory)
  - 5 questions on Relations and Functions
    - 3 Relations/Functions, 2 Counting
  - Bring: 1-page handwritten cheat-sheet, calculator, scratch papers
  - No reentry after leaving
  - Study with example problems BOTH in Lec Notes and HWs. For more look into the TextBook
  - MORE DIFFICULT THAN MT1!!!
- This Lecture
  - Review for Midterm #2: Hw 4 & 5

Homework 4
(Total 25 pts)

CSC230 Discrete Math
Kazunori Okada
HW#4 Q1 (4pt)

Determine whether the binary relation $R$ on the set of all people is reflexive, symmetric, antisymmetric, transitive where $(a, b) \in R$ if and only if: (i) $a$ is taller than $b$

- **ANS**
  
  $\forall a \in S, (a,a) \in R$?  $(a,a) \in R$
  
  *not reflexive*  
  *not symmetric*  
  *antisymmetric*  
  *transitive*  
  
  $(a,b) \in R$  
  
  *not taller than my self*  
  *if $a$ is taller than $b$ the $b$ is not taller than $a$*  
  *YES*  
  *YES*  
  *YES*  
  *YES*  
  
  if $(a,b) \in R$ then $(b,a) \not\in R$

HW#4 Q1 (4pt)

Determine whether the relation $R$ on the set of all people is reflexive, symmetric, antisymmetric, transitive where $(a,b) \in R$ if and only if: (ii) $a$ has the same last name as $b$

- **ANS**
  
  reflexive : YES
  
  symmetric : YES
  
  antisymmetric : NO
  
  transitive : YES

Equivalence Relation
Let R be the relation on the set \{0,1,2,3\} with ordered pairs (0,1),(1,1),(1,2),(2,0),(2,2) and (3,0). Find the (i) reflexive closure of R

\[\text{Reflexive Closure} \rightarrow R \cup \text{Id} = \{(0,0), (1,1), (2,2), (3,3)\}\]

\[R^+ = R \cup R^2 \cup R^3 \cup \cdots \cup R^\infty\]

\[\text{Let } R^+ \text{ be reflexive and transitive} \rightarrow R^+ = \text{Id} \cup R^+

Let R be the relation on the set \{0,1,2,3\} with ordered pairs (0,1),(1,1),(1,2),(2,0),(2,2) and (3,0). Find the (ii) symmetric closure of R

\[\text{Symmetric Closure} \rightarrow R \cup R^{-1} = \{(0,1), (1,1), (1,2), (2,0), (2,2), (3,3)\}\]
Let $R$ be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad=bc$. (i) Show that $R$ is an equivalence relation.

- **ANS**

  Reflexive: $(a,b),(a,b) \in R \iff a \cdot b = b \cdot a$.
  
  Symmetric: if $(a,b),(c,d) \in R$ then $ad=bc \iff cb=da$.
  
  Transitive: $(a,b),(c,d) \in R$ and $(c,d),(e,f) \in R$ \iff $(a,b),(e,f) \in R$.

\begin{align*}
&\iff \quad a \cdot d = b \cdot c = a \cdot f = d \cdot e \\
&\iff \quad acdf = bcde \quad \text{since } c>0, d>0 \\
&\iff \quad (a,b),(e,f) \in R.
\end{align*}

Let $R$ be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad=bc$. (ii) What is $[(1,2)]$, equivalence class of $(1,2)$?

- **ANS**

  $(1,2),(c,d) \in R \iff d = 2c$, $c>0$.

  $[(1,2)] = \{ \text{any pair } (c,d) \text{ with } \frac{d}{c} = 2 \}$

  or any pair $(c,2c)$ \quad $c \in \mathbb{N}^+$

  or $\{ (1,2),(2,4),(3,6),(4,8),(5,10),\ldots \}$

  $\{ (c,d) \mid d=2c, \ c \in \mathbb{N}^+ \}$
Homework 5
(Total 25 pts)

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HW#5 Q1 (2pt)

Determine whether each of these functions is bijection from $\mathbb{R}$ to $\mathbb{R}$ (explain your solutions) (a) $f(x) = -3x + 4$

- ANS
  
  To show $f(x)$ is bijection, we have to show that $f$ is both 1-1 and Onto.

  (1-1) For all $a, b \in \mathbb{R}$, if $f(a) = f(b)$ then $a = b$.

  \[ f(a) = f(b) \quad \Rightarrow \quad -3a + 4 = -3b + 4 \quad \Rightarrow \quad a = b \]

  (Onto) For all $b \in \mathbb{R}$ (codomain), there exists $a \in \mathbb{R}$ such that $f(a) = b$.

  \[ b = f(a) = -3a + 4 \quad \Rightarrow \quad a = \frac{4 - b}{3} \quad \text{(domain)} \]

  Since $b \in \mathbb{R}$, $a \in \mathbb{R}$ (domain). Thus $f(x)$ is onto.
HW#5 Q1 (2pt)

Determine whether each of these functions is bijection from \( \mathbb{R} \) to \( \mathbb{R} \) (explain your solutions) (b) \( f(x) = \frac{x+1}{x+2} \)

- ANS

\[ f(-2) \text{ is undefined} \]

\[ x = -2 \implies \frac{-2+1}{-2+2} = \frac{1}{0} \text{ is undefined} \]

\[ x = -2 \notin \mathbb{R} \]

\[ \therefore \text{Not bijection, not a function} \]

\[ \therefore \quad x + 2 = x + 1 \]

HW#5 Q2 (1.5pt)

Let \( f(x) = x^2 + 1, \ g(x) = x + 2 \) are functions from \( \mathbb{R} \) to \( \mathbb{R} \). Find (i) \( f \circ g \)

- ANS

\[ (a,b) \quad (b,c) \]

\[ (a,c) \]

\[ f \circ g = f(g(x)) = (x+2)^2 + 1 \]

\[ = x^2 + 4x + 5 \]

\[ f(1) = \{ (0,1), (2,1), (3,1) \} \]

\[ S(f) = \{ \} \quad \text{and} \quad \{3\} \]
HW#5 Q2 (1.5pt)

Let $f(x) = x^2 + 1$, $g(x) = x + 2$ are functions from $\mathbb{R}^\times \mathbb{R}$.

Find (ii) $g \circ f$

- ANS
  
  $g \circ f = g(f(x)) = (x^2+1)+2$
  
  $= x^2+3$

HW#5 Q2 (1.5pt)

Let $f(x) = x^2 + 1$, $g(x) = x + 2$ are functions from $\mathbb{R}^\times \mathbb{R}$.

Find (iii) $f + g$

- ANS
  
  $f + g = f(x) + g(x)$
  
  $= x^2 + 1 + x + 2$
  
  $= x^2 + x + 3$

  $f = \{ (a, b) \mid a, b \in \mathbb{R} \}$
  $g = \{ (c, d) \mid c, d \in \mathbb{R} \}$
  
  $f + g = \{ (a, b) + (c, d) \mid a, b, c, d \in \mathbb{R} \}$
HW#5 Q2 (1.5pt)

Let \( f(x) = x^2 + 1 \), \( g(x) = x + 2 \) are functions from \( \mathbb{R} \times \mathbb{R} \).

Find (iv) \( fg \)

\[
fg = f(g(x)) = (x^2 + 1)(x + 2) = x^3 + x + 2x^2 + 2 = \frac{3}{2}x^2 + x + 2
\]

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HW#5 Q4 (2pt)

Find values of these summations: (i) \( \sum_{i=1}^{3} \sum_{j=1}^{2} (i-j) \)

\[
= \sum_{i=1}^{3} \left[ \sum_{j=1}^{2} (i-j) \right]
= \sum_{i=1}^{3} \left[ (i-1) + (i-2) \right]
= 2 + 1 + 3 + 1 + 0 + 0 = 6
\]

\[
\sum_{i=1}^{3} (2i-1) = 3 \cdot \frac{3}{2} \cdot 3 = \frac{27}{2}
\]

\[
\sum_{i=1}^{3} 3i = 9 + 6 + 3 = 18
\]

\[
\sum_{i=1}^{3} \frac{1 + 3}{2} = \frac{4}{2} + \frac{4}{2} + \frac{4}{2} = 3
\]

\[
\sum_{i=1}^{3} i^2 = \frac{3(3+1)(2*3+1)}{6} = 14
\]

\[
\sum_{i=1}^{3} \frac{1 + 3}{2} = \frac{4}{2} + \frac{4}{2} + \frac{4}{2} = 3
\]

\[
\sum_{i=1}^{3} i^3 = \left( \frac{3(3+1)(2*3+1)}{6} \right)^2 = 36
\]

\[
\sum_{i=1}^{3} \frac{1 + 3}{2} = \frac{4}{2} + \frac{4}{2} + \frac{4}{2} = 3
\]

\[
\sum_{i=1}^{3} i^3 = \left( \frac{3(3+1)(2*3+1)}{6} \right)^2 = 36
\]
HW#5 Q4 (2pt)

Find values of these summations: (ii) \(\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3\)

- ANS
  \[
  \sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3 = \sum_{i=0}^{2} \left( i^2 \sum_{j=0}^{3} j^3 \right) = \sum_{i=0}^{2} (i^2 \cdot 2) = 2(0^2 + 1^2 + 2^2 + 3^2) = 2(0 + 1 + 4 + 9) = 2 \cdot 14 = 28.
  \]

HW#5 Q5 (3pt)

Determine whether each of these sets is countable or uncountable. Explain your solutions. (i) the odd negative integers

- ANS

Let \(X = \{ x | x \in \mathbb{Z}^-, x = 2n+1 \text{ for some } n \in \mathbb{Z} \}\).

Let \(f : \mathbb{N} \rightarrow X\) be defined as \(f(a) = \frac{2a-1}{2}\) for \(a \in \mathbb{N}\).

Then \(f(a)\) can be defined as \(2n+1\) for \(n \geq 0\).

Thus \(f(a)\) is a one-to-one correspondence and \(f\) is a bijection.

\(f(0) = 1, f(1) = 3, f(2) = 5, \ldots\)
HW#5 Q5 (3pt)

Determine whether each of these sets is countable or uncountable. Explain your solutions. (ii) the real number between 1 and 2

• ANS

Use the exactly same argument for the example in the last two slides of the function lecture except that using "1,...," instead of "0,...", \( x \in \{x|0<x<1\} \).

1) Suppose \( Y \) is countable...
   \[ Y = \{x|0<x<1\} \]

2) Then you can list all elements in \( Y \) in an order...

3) Thus define \( r^* \) that is different from \( \forall c \in Y \) at all.

4) \( r^* \) & \( Y \) but \( r^* \notin \) \( Y \) because \( 1 < r^* < 2 \) contradicts.

HW#5 Q8 (2.5pt)

Suppose that there are nine students in a class. (a) Show that the class must have at least five male students or at least five female students

• ANS

by generalized pigeonhole principle, \( \left\lceil \frac{9}{2} \right\rceil = \left\lceil 4.5 \right\rceil = 5 \)

at least 5 (i.e. \( \left\lceil 9/2 \right\rceil \) where 2 comes from the choice of gender) must be either male or female.
Suppose that there are nine students in a class. (b) Show that the class must have at least three male students or at least seven female students.

• ANS

Proof by contradiction.

\( \neg(P \lor Q) \)

\( \Rightarrow \) Class does not have at least 3 male students and at least 7 female students.

\( \Rightarrow \) Class has at most 2 male and Class has at most 6 female.

\( \Rightarrow \) Class has at most 8 students.

\( \Rightarrow \) Contradiction.

\( \square \)