Overview

• MT2 will be on Nov 9, on relations, functions and algorithms; Start to study for it! Make hand-written notes while at it.
• HW#5 due on Oct 21. Work on it!

• Last Lecture
  – Function: Definition: A special type of binary relation!!!
  – Function Properties: 1-1, Onto, 1-1 Correspondences
  – Function Operations: Composition.
• This Lecture: Complete Functions by the next two lectures.
  – Function Operations: Composition, Inverse, Sum, Product
  – Sequence, Subsequence, Summation
  – Theory of Counting
  – Pigeonhole Principle
  – Generalized Pigeonhole Principle

Chapter 4. Functions
Cond.
Composition of functions:

**Definition:** Let $f: S \to T$ and $g: T \to U$. Then the composition function, $g \circ f$, is a function from $S$ to $U$ defined by $(g \circ f)(s) = g(f(s))$.

**Note 1:** This is same as the composition of relations.

**Note 2:** The function $g \circ f$ is applied from right to left; function $f$ is applied first and then function $g$.

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**Example:**

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.

Let $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \lfloor x \rfloor$.

Note: The floor function $\lfloor x \rfloor$ associates with each real number $x$ the greatest integer less than or equal to $x$ (see previous examples).

What is the value of $(g \circ f)(2.5)$?

$(g \circ f)(2.5) = g(f(2.5)) = g(2.5^2) = g(6.25) = \lfloor 6.25 \rfloor = 6$

What is the value of $(f \circ g)(2.5)$?

$(f \circ g)(2.5) = f(g(2.5)) = f(\lfloor 2.5 \rfloor) = f(2) = 2^2 = 4$
Theorem: Let \( f: X \to Y \) and \( g: Y \to Z \) be functions

- If \( f \) and \( g \) are both 1-1, then \( g \circ f \) is 1-1
- If \( f \) and \( g \) are both onto, then \( g \circ f \) is onto
- If \( f \) and \( g \) are both 1-1 correspondences, then \( g \circ f \) is a 1-1 correspondence
- If \( g \circ f \) is 1-1, then \( f \) is 1-1
- If \( g \circ f \) is onto, then \( g \) is onto

Operations: Inverse of Functions & Identity Function

Definition: Let \( f = \{ (x, y) \in A \times B : f(x) = y \} \). The inverse of \( f \), denoted by \( f^{-1} \),

\[
f^{-1} = \{ (y, x) \in B \times A : f(y) = x \}.
\]

Theorem: Let \( f: S \to T \). Then \( f \) is a bijection (1-1 correspondence) if and only if \( f^{-1} \) exists.

Definition: The function that maps each element of a set \( S \) to itself is called the identity function on \( S \), i.e. \( i(x) = x \)

Theorem: Composition of a bijective function and its inverse:

\[
(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x
\]

i.e. The composition of a function and its inverse is the identity function \( i(x) = x \).
Operations: Sum & Product of Functions

Let \( f_1 \) and \( f_2 \) be functions from \( A \) to \( \mathbb{R} \). Then the sum and the product of \( f_1 \) and \( f_2 \) are also functions from \( A \) to \( \mathbb{R} \) defined by:

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x)
\]

\[
(f_1 f_2)(x) = f_1(x) f_2(x)
\]

Example:

\( f_1(x) = 3x, \quad f_2(x) = x + 5 \)

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5
\]

Let \( x = 10 \),\( f_1(10) + f_2(10)=30+15=45 \)  \( (f_1 + f_2)(10) =40+5=45 \)

\( (f_1 f_2)(x) = f_1(x) f_2(x) = 3x (x + 5) = 3x^2 + 15x \)

Let \( x = 10 \),  \( f_1(10) * f_2(10)=30*15=450 \)

\( (f_1 * f_2)(10) =300+150=450 \)

Sequence: list of objects

- Definition:

  A sequence lists up/order things in the set \( S \).
Sequence: list of objects

- **Definition:** A sequence is a function from a subset of integers (usually \{0,1,2\ldots\} or \{1,2,3\ldots\}) to a set \(S\). A sequence lists up/order things in the set \(S\). A set does not regard an order of elements by definition! But sequence does.

- **Example 1:** a simple sequence “a,b,c” is a sequence with 3 terms (finite sequence) “a,c,b”?
  \(f=\{(1,a),(2,b),(3,c)\} \rightarrow f: \mathbb{N} \rightarrow S\)

- **Example 2:** a geometric progression is a sequence:
  \(a, ar, ar^2, ar^3, \ldots\) where \(a\) and \(r\) are real numbers, i.e. \(f(n)=ar^n\), for \(n \in \mathbb{N}\) (infinite sequence)

- **Definition:** A subsequence is a shorter sequence with the same order as they do in the original sequence.

- **Example 3:** Use sequence \(X = 1,2,3,5,8,13,21\)
  \(\{1,1\}, \{(1,2),(2,3),(3,5)\},\{(5,8),(8,13),(13,21)\}\)

  - \(1,3,13,21\) is a subsequence of \(X\)
  - \(1,8,5\) is not a subsequence of \(X\)
  - \(1,2,3,5,8,13,21\) is a subsequence of \(X\)
• Definition: Given a sequence \(a_m, a_{m+1}, \ldots, a_n\). We use the notation \(\sum_{j=m}^n a_j\) to represent summation \(a_m + a_{m+1} + \ldots + a_n\).

• Example: \(\sum_{j=1}^n j = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}\)

Let \(x = 1+2+3+\ldots+n\)

\(\Rightarrow x = n(n-1)+\ldots+1\)

\[2x = n(n+1)\]

\[x = \frac{n(n+1)}{2}\]

• Example: \(\sum_{i=0}^n ar^i, (r \neq 0) = \frac{ar^{n+1} - a}{r-1}, r \neq 1\)

• Let \(x = a + ar + ar^2 + \ldots + ar^n\)

\(\Rightarrow xr = ar + ar^2 + \ldots + ar^{n+1}\)

\[x(r-1) = (ar^{n+1} - a)\]

\[\sum_{i=1}^5 \sum_{j=1}^2 ij = \sum_{i=1}^5 (i+2i) = \sum_{i=1}^5 3i = 3+6+9+12+15 = 45\]

Double Summations

Corresponding to a double loops in C or Java, there is also triple and quadruple summation corresponding to nested loops:

\[\sum_{x=a}^b \sum f(x) = f(a) + f(a+1) + \cdots + f(b)+f(b)\]

\[\sum_{i=1}^5 \sum_{j=1}^2 ij = 3+6+9+12+15 = 45\]
Theory of Counting

• Review: Given a function \( f: S \rightarrow T \)

  The range \( R = f(S) = \{ f(s) \mid s \in S \} \subseteq T \).

  Then \( |R| \leq |T| \) \( \text{note : } |X| \text{ is cardinality of } X \)

If \( f \) is injection, 1-1, then \( |S| = |R| \leq |T| \)

For all \( a, b \) in \( S \), if \( f(a) = f(b) \), then \( a = b \)

If \( f \) is surjection, onto, then \( |S| \geq |R| = |T| \)

For all \( b \) in \( T \), we have \( a = f^{-1}(b) \) in \( S \)

If \( f \) is bijection, 1-1 correspondence, then \( |S| = |R| = |T| \)

\( \iff |S| = |R| \leq |T| \) \( \text{and at the same time} \)

\( \iff |S| \geq |P| = |T| \)
Overview

- MT2 in 3 weeks on Nov 9, on relations, functions and algorithms; Make a well-organized hand-written notes.
- HW#5 due in two days on Oct 21. Work on it!

Last Lecture
- Function: Definition: A special type of binary relation!!!
- Function Properties: 1-1, Onto, 1-1 Correspondences
- Function Operations: Composition.
- Function Operations: Inverse, Sum, Product
- Sequence, Subsequence, Summation
- Theory of Counting

This Lecture: Complete Functions
- Theory of Counting cond
- Pigeonhole Principle
- Generalized Pigeonhole Principle

• Review: Let $X$ be a set and $n \in \mathbb{N}$. If $|X| = |\{0,1,2,\ldots,n-1\}|$, then the cardinality of $X$ is $n$ and $X$ is finite. We say $X$ is infinite if $X$ is not finite
• Definition: Let $X$ be a set. $X$ is countably infinite if $|X| = |\mathbb{N}| = \infty$.

$X$ is countable if it is either finite or countably infinite. If a set is not countable, then it is uncountable.

Theorem: If we can define a bijective function $f: \mathbb{N} \rightarrow X$, then $|\mathbb{N}| = |X|$ thus a set $X$ is countably infinite
• Example: \( X = \{ n \in \mathbb{N} \text{ and } n \text{ is even} \} \) is countably infinite.
  
  \[ f: \mathbb{N} \to X \text{ where } f(n) = 2n \]
  
  \( f \) is 1-1 correspondence (prove it!)

• Example: Integer set \( \mathbb{Z} = \{ ...,-2,-1,0,1,2,... \} \) is countably infinite.
  
  Let us list \( \mathbb{Z} \) as \( 0, -1, 1, -2, 2, -3, 3, ... \)
  
  then define \( f: \mathbb{N} \to \mathbb{Z} \) where
  
  \[ f(0) = 0, f(2n) = n, f(2n-1) = -n \text{ for } n > 0 \]
  
  \( f \) is 1-1 correspondence (prove it!)

• Example: \( \mathbb{Q}^+ = \) set of all positive rational numbers, \( \{ m/n : m \in \mathbb{Z}, n \in \mathbb{Z}, n > 0 \} \), is countably infinite?

Find a way to list all members of \( \mathbb{Q}^+ \) as an infinite sequence and each real number is only listed once. Therefore, \( \mathbb{N} \to \mathbb{Q}^+ \) is 1-1 correspondence (bijection).

Use following argument, list the positive rational numbers \( p/q \) with \( p + q = 2 \), followed by those with \( p + q = 3 \), then \( p + q = 4 \), etc. If a number already listed, do not list them again.

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The first few terms are 1, 1/2, 2, 3, 1/3, 1/4, 1/5, 1/6, 2/3, 3/2, 4, 5...
• Example: $\mathbb{Q} = \text{set of all rational numbers}$ is countably infinite. List all positive and negative rational numbers using previous examples for $\mathbb{Z}$ and $\mathbb{Q}^+$. 

Example: Real numbers set $\mathbb{R}$ is uncountable. We only need to show that the subset $X$ of all real numbers $\mathbb{R}$ between $0$ and $1$ is uncountable. (If a subset of $\mathbb{R}$ is uncountable, then $\mathbb{R}$ must be uncountable!!!) $\leftarrow$ Cantor diagonalization $X \subseteq \mathbb{R}$ so $|X| \leq |\mathbb{R}|$ !

Suppose that $X$ is countable and then arrive a contradiction.
Suppose $X = \{x | x \in \mathbb{R} \text{ and } 0 < x < 1\}$ is countable, then we can list all real numbers between $0 < x < 1$. Let $r_i \in X$ for all $i = 1, 2, 3, \ldots$

\[
\begin{align*}
  r_1 &= 0.d_{11}d_{12}d_{13} \ldots \\
  r_2 &= 0.d_{21}d_{22}d_{23} \ldots \\
  r_3 &= 0.d_{31}d_{32}d_{33} \ldots \\
  \vdots \\
  r_i &= 0.d_{i1}d_{i2}d_{i3} \ldots d_{ii} \ldots \\
  \vdots 
\end{align*}
\]

Note: $d_{xy} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Now, let’s construct a new real number $r^*$ s.t. $0 < r^* < 1$ as follows:

\[
r^* = 0. d_1d_2d_3 \ldots d_i \ldots \text{ for all } i = 1, 2, 3, \ldots
\]

such that if $d_{ii} \neq 4$, then $d_i = 4$; otherwise $d_i = 5$.

Clearly, $r^* \notin X$ since for each $r_i \in X$, $d_i \neq d_{ii}$, however we supposed $0 < r^* < 1$. So this is a contradiction.

Therefore, all the real numbers between $0$ and $1$ cannot be listed thus uncountable. Then $\mathbb{R}$ is uncountable because $\mathbb{R}$ contains more elements than $X$. 

Example:

\[
\begin{align*}
  r_1 &= 0.37941 \ldots \\
  r_2 &= 0.45003 \ldots \\
  r_3 &= 0.06078 \ldots \\
  r_4 &= 0.95679 \ldots \\
  r_5 &= 0.24531 \ldots \\
  \vdots \\
  r^* &= 0.4545454 \ldots 
\end{align*}
\]
**Pigeonhole Principle:** If \( x \) items are placed into \( y \) bins where \( x > y \), then there is one bin which contains at least two items.

Note: textbook specified this property by using function notation

Proof by contradiction: Suppose \( x (> y) \) items are placed into \( y \) bins and suppose no bin with two or more items. Each bin has 0 or 1 item. The maximum total number of items in \( y \) bins is then \( y \), but \( y < x \). This contradiction proves the result

Example: How many people must be in a room to guarantee that two people have last names that begin with the same letter?

There are 26 letters (or bins). If there are 27 people, then at least 2 people will have last names beginning with the same letter.

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Example: The population of city \( x \) is about 40,000. If each resident has three initials, is it true that there must be at least 2 individuals with the same initials?

By Pigeonhole principle, there must be at least 2 individuals with the same initials

**Generalized Pigeonhole Principle:** If \( x \) items are placed into \( y \) bins where \( x > y \), then there is one bin which contains at least \( \lceil x/y \rceil \) items. (note: ceiling)

Example: For above example, it is true that there must be at least \( \lceil 40000/17576 \rceil = 3 \) individual with the same initials, i.e \( \lceil 40000/17576 \rceil = 3 \).
Example: What is the minimum number of people in a group there must be so that there are at least 3 who were born in the same month? There are 12 months (bins). With > 12 people, at least 2 people who were born in the same month. With > 24 people, at least 3 people who were born in the same month. So, minimum number is 25 people.

Example: Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

Ans: ??

Application - Problem with divisors:
Let \( m \in \mathbb{N} \). Given \( m \) integers \( a_1, a_2, \ldots, a_m \), there exist \( i \) and \( j \) with \( 0 < i < j \leq m \) such that \( a_{i+1} + a_{i+2} + \ldots + a_j \) is divisible by \( m \).
Proof (in two cases):
• Consider $m$ sums:
  \[ a_1, \ a_1 + a_2, \ a_1 + a_2 + a_3, \ \ldots, \ a_1 + a_2 + \ldots + a_m \]
• 1) If any of these sums is divisible by $m$, then we are done!
• 2) Suppose not, each sum has a nonzero remainder when divided by $m$.
• The possible remainders are $1, 2, 3, \ldots, m-1$.
• By Pigeonhole principle, there are at least $2$ sums with same remainder $(r)$. We have :
  \[ a_1 + a_2 + \ldots + a_i = cm + r \] \hspace{1cm} (i) \\
  \[ a_1 + a_2 + \ldots + a_j = dm + r \] \hspace{1cm} (ii)
where $c, d$ and $r$ are integers and assume $i < j$
•  \[ a_{i+1} + a_{i+1} + \ldots + a_j = (d-c) m \] \hspace{1cm} (ii) – (i)
It is divisible by $m$. 
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