Overview

• Have a nice break!!!
• MT2 will be on Apr 20, on relations, functions and algorithms; Start to study for it! Make hand-written notes while at it.
• HW#5 due after the break on Apr 1. Work on it!
• Last Lecture
  – Function: Definition: A special type of binary relation!!!
  – Function Properties: 1-1, Onto, 1-1 Correspondences
  – Function Operations: Composition.
• This Lecture: Complete Functions by the next two lectures.
  – Function Operations: Inverse, Sum, Product
  – Sequence, Subsequence, Summation
  – Theory of Counting
  – Pigeonhole Principle
  – Generalized Pigeonhole Principle

Chapter 4. Functions
Cond.
Operations: Inverse of Functions & Identity Function

Definition: Let \( f = \{(x,y) \in A \times B : f(x) = y\} \). The **inverse** of \( f \), denoted by \( f^{-1} \),

\[
 f^{-1} = \{(y,x) \in B \times A : f(y) = x\}. 
\]

**Theorem:** Let \( f: S \to T \). Then \( f \) is a bijection (1-1 correspondence) if and only if \( f^{-1} \) exists.

Definition: The function that maps each element of a set \( S \) to itself is called the **identity function on \( S \)**, i.e. \( i(x) = x \)

**Theorem:** Composition of a bijective function and its inverse:

\[
 (f^{-1} \circ f)(x) = f^{-1}(f(x)) = x 
\]

i.e. The composition of a function and its inverse is the identity function \( i(x) = x \).

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Operations: Sum & Product of Functions

Let \( f_1 \) and \( f_2 \) be functions from \( A \) to \( \mathbb{R} \). Then the **sum** and the **product** of \( f_1 \) and \( f_2 \) are also functions from \( A \) to \( \mathbb{R} \) defined by:

\[
 f_1 + f_2 (x) = f_1(x) + f_2(x) \\
 f_1 \cdot f_2 (x) = f_1(x) \cdot f_2(x) 
\]

**Example:**

\[
 f_1(x) = 3x, \quad f_2(x) = x + 5 \\
 (f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5 \\
 (f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) = 3x(x + 5) = 3x^2 + 15x \\
 \text{Let } x = 10, \quad f_1(10) + f_2(10) = 30 + 15 = 45 \quad (f_1 + f_2)(10) = 40 + 5 = 45 \\
 (f_1 \cdot f_2)(10) = 30 \cdot 15 = 450 \\
 \text{Let } x = 10, \quad f_1(10) \cdot f_2(10) = 30 \cdot 15 = 450 \quad (f_1 \cdot f_2)(10) = 300 + 150 = 450 
\]
Sequence: list of objects

- Definition: A sequence lists up/order things in the set $S$.

  A sequence lists up/order things in the set $S$. A set does not regard an order of elements by definition! But sequence does.

- Example 1: a simple sequence “$a, b, c$” is a sequence with 3 terms (finite sequence) “$a, c, b$”?

  $$f=\{(1, a), (2, b), (3, c)\} \rightarrow f: \mathbb{N} \rightarrow S \quad \mathbb{N} = \{1, 2, 3\}$$

- Example 2: a geometric progression is a sequence: $a, ar, ar^2, ar^3, \ldots$ where $a$ and $r$ are real numbers i.e. $f(n)=ar^n$, for $n \in \mathbb{N}$ (infinite sequence) $f: \mathbb{N} \rightarrow S$. 

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• Definition: A **subsequence** is a shorter or the same length sequence with the same order as they do in the original sequence.

• Example 3: Use sequence \( X = 1,2,3,5,8,13,21 \) \( \{1, 2, 3, 3, 6, 8, 13, 21\} \)

1,3,13,21 is a subsequence of \( X \)  
1,8,5 is not a subsequence of \( X \)  
1,2,3,5,8,13,21 is a subsequence of \( X \)

• Definition: Given a sequence \( a_m, a_{m+1}, \ldots, a_n \). We use the notation \( \sum_{j=m}^{n} a_j \) to represent **sum** \( a_m + a_{m+1} + \ldots + a_n \).

• Example: \( \sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \)

Let \( x = 1 + 2 + 3 + \cdots + n \)

\( + x = n + (n-1) + \cdots + 1 \)

\( 2x = n(n+1) \)

\( x = \frac{n(n+1)}{2} \)

• Example: \( \sum_{k=0}^{n} a^k, (r \neq 0) = \frac{a^{n+1} - a}{r - 1}, r \neq 1 \)

• Let \( x = a + ar + ar^2 + \cdots + ar^n \)

\( \rightarrow \) \( xr = ar + ar^2 + \cdots + ar^{n+1} \)

\( x(r-1) = (ar^{n+1} - a) \)
Double Summations

Corresponding to a double loops in C or Java, there is also triple and quadruple summation corresponding to nested loops:

\[ \sum_{i=1}^{b} f(i) = f(1) + f(2) + \cdots + f(b) \]

**Example:**

\[ \sum_{i=1}^{5} i = \sum_{i=1}^{5} (i + 2i) = \sum_{i=1}^{5} 3i = 3 + 6 + 9 + 12 + 15 = 45 \]

**Triple?**

\[ \sum_{i=1}^{5} \sum_{j=1}^{2} ij = \sum_{i=1}^{5} (i + 2i) = \sum_{i=1}^{5} 3i = 3 + 6 + 9 + 12 + 15 = 45 \]

**Theory of Counting**

Let's think about the size of sets when you have an infinite set \( A \)

\[ |A| = \infty \]

**Cardinality**

**Theory of Counting**

A \( \rightarrow |A| = \infty \)

B = add 1 new elements to A

C = add 1000 new elements to A

Which one is longer?

\[ |A| < |B| < |C| ? \]

No

\[ |A| = |B| = |C| = \infty \]

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Review: Given a function \( f: S \rightarrow T \)

The range \( R = f(S) = \{ f(s) \mid s \in S \} \subseteq T \).

Then \(|R| \leq |T|\) note: \(|X|\) is cardinality of \(X\)

If \(f\) is injection, 1-1, then \(|S| = |R| \leq |T|\)

For all \(a, b \in S\), if \(f(a) = f(b)\), then \(a = b\)

If \(f\) is surjection, onto, then \(|S| \geq |R| = |T|\)

For all \(b \in T\), we have \(a = f^{-1}(b) \in S\)

If \(f\) is bijection, 1-1 correspondence, then \(|S| = |R| = |T|\)

\(\Leftrightarrow\) both injection \& surjection

\(|S| \geq |R| \leq |T|\) at the same time

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**Overview**

- **MT2 in 3 weeks on Apr 20, on relations, functions and algorithms;** Make a well-organized hand-written notes.
- **HW#5 due in two days on Apr 1. Work on it!**
- **Last Lecture**
  - Function: Definition: A special type of binary relation!!!
  - Function Properties: 1-1, Onto, 1-1 Correspondences
  - Function Operations: Composition.
  - Function Operations: Inverse, Sum, Product
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  - Theory of Counting
- **This Lecture: Complete Functions**
  - Theory of Counting cond
  - Pigeonhole Principle
  - Generalized Pigeonhole Principle

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• Review: Let $X$ be a set and $n \in \mathbb{N}$. If $|X| = |\{0,1,2,\ldots,n-1\}|$, then the cardinality of $X$ is $n$ and $X$ is finite. We say $X$ is infinite if $X$ is not finite.

• Definition: Let $X$ be a set. $X$ is countably infinite if $|X| = |\mathbb{N}| = \infty$.

• $X$ is countable if it is either finite or countably infinite. If a set is not countable, then it is uncountable.

• **Theorem:** If we can define a bijective function $f : \mathbb{N} \to X$, then $|\mathbb{N}| = |X|$ thus a set $X$ is countably infinite.

To show $X$ is countable...
1. Find a function that associates all elements in $X$ with $\mathbb{N}$. Then prove it is bijective.
2. Show that there is a way to list all elements in $X$ without repeats and missing any.

Why?

• Example: $X = \{n \in \mathbb{N} \text{ and } n \text{ is even}\}$ is countably infinite. $f : \mathbb{N} \to X$ where $f(n) = 2n$.

$f$ is 1-1 correspondence (prove it!)

• Example: Integer set $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$ is countably infinite.

Let us list $\mathbb{Z}$ as $0$, $-1$, $1$, $-2$, $2$, $-3$, $3$, $\ldots$

then define $f : \mathbb{N} \to \mathbb{Z}$ where
$f(0) = 0$, $f(2n) = n$, $f(2n-1) = -n$ for $n > 0$

$f$ is 1-1 correspondence (prove it!)

• Example: $\mathbb{Q}^+ = \text{set of all positive rational numbers}$, $\{m/n : m \in \mathbb{Z}, n \in \mathbb{Z}, n > 0\}$, is countably infinite.
• Find a way to list all members of $\mathbb{Q}^+$ as an infinite sequence and each real number is only listed once. Therefore, $\mathbb{N} \rightarrow \mathbb{Q}^+$ is 1-1 correspondence (bijection).

• Use following argument, list the positive rational numbers $p/q$ with $p+q=2$, followed by those with $p+q=3$, then $p+q=4$, etc. If a number already listed, do not list them again.

| 1 | 1/1 | 1/2 | 1/3 | 1/4 | 1/5 | 1/6 | 1/7 | 1/8 | ...
|---|-----|-----|-----|-----|-----|-----|-----|-----|---
| 2 | 2/1 | 2/3 | 2/5 | 2/7 | 2/9 | 2/11 | 2/13 | 2/15 | ...
| 3 | 3/1 | 3/2 | 3/4 | 3/5 | 3/7 | 3/8 | 3/10 | 3/12 | ...
| 4 | 4/1 | 4/3 | 4/5 | 4/7 | 4/9 | 4/11 | 4/13 | 4/15 | ...
| 5 | 5/1 | 5/2 | 5/3 | 5/4 | 5/6 | 5/7 | 5/8 | 5/9 | ...

The first few terms are $1, 1/2, 2, 3, 1/3, 2/3, 3/2, 4, 5, ...$

Example: $\mathbb{Q} = \text{set of all rational numbers}$ is countably infinite.

List all positive and negative rational numbers using previous examples for $\mathbb{Z}$ and $\mathbb{Q}^+$.

Example: Real numbers set $\mathbb{R}$ is uncountable. We only need to show that the subset $X$ of all real numbers $\mathbb{R}$ between 0 and 1 is uncountable. (If a subset of $\mathbb{R}$ is uncountable, then $\mathbb{R}$ must be uncountable!!) \( \Leftarrow \) Cantor diagonalization $X \subseteq \mathbb{R}$ so $|X| \leq |\mathbb{R}|$!

Suppose that $X$ is countable and then arrive a contradiction.

Suppose $X = \{x | x \in \mathbb{R} \text{ and } 0 < x < 1\}$ is countable, then we can list all real numbers between $0 < x < 1$. Let $r_i \in X$ for all $i = 1, 2, 3, ...$

Let $r_i = 0, d_{i1} d_{i2} d_{i3} ...$

Note: $d_{xy} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Now, let’s construct a new real number $r^*$ s.t. $0 < r^* < 1$ as follows:

$$r^* = 0. \overline{d_1 d_2 d_3 \ldots d_i \ldots}$$

for all $i = 1, 2, 3, \ldots$ such that if $d_{ii} \neq 4$, then $d_i = 4$; otherwise $d_i = 5$.

Clearly, $r^* \notin X$ since for each $r_i \in X$, $d_i \neq d_{ii}$, however we supposed $0 < r^* < 1$. So this is a contradiction.

Therefore, all the real numbers between 0 and 1 cannot be listed thus uncountable. Then $\mathbb{R}$ is uncountable because $\mathbb{R}$ contains more elements than $X$.

Example:

$$r_1 = 0.37941\ldots$$
$$r_2 = 0.45039\ldots$$
$$r_3 = 0.6187\ldots$$
$$r_4 = 0.579\ldots$$
$$r_5 = 0.2456\ldots$$
$$\ldots$$

$$r^* = 0.45454\ldots$$

Pigeonhole Principle: If $x$ items are placed into $y$ bins where $x > y$, then there is one bin which contains at least two items.

Proof by contradiction: Suppose $x (> y)$ items are placed into $y$ bins and suppose no bin with two or more items. Each bin has 0 or 1 item. The maximum total number of items in $y$ bins is then $y$, but $y < x$.

This contradiction proves the result

Example: How many people must be in a room to guarantee that two people have last names that begin with the same letter?

There are 26 letters(or bins). If there are 27 people, then at least 2 people will have last names beginning with the same letter.

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Example: The population of city x is about 40,000. If each resident has three initials, is it true that there must be at least 2 individuals with the same initials?

How many possible combination of 3 letters?
\[26 \times 26 \times 26 = 17,576 < 40,000\]

By Pigeonhole principle, there must be at least 2 individual with the same initials.

**Generalized Pigeonhole Principle:** If \(x\) items are placed into \(y\) bins where \(x > y\), then there is one bin which contains at least \(\lceil x/y \rceil\) (note: ceiling) items.

Example: For above example, it is true that there must be at least 3 individual with the same initials, i.e \(\lceil 40000/17576 \rceil = 3\).

Example: What is the minimum number of people in a group there must be so that there are at least 3 who were born in the same month?

There are 12 months (bins). With > 12 people, at least 2 people who were born in the same month.

With > 24 people, at least 3 people who were born in the same month.

So, minimum number is 25 people.

Example: Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

Ans: ??
Application - Problem with divisors:

Let \( m \in \mathbb{N} \). Given \( m \) integers \( a_1, a_2, ..., a_m \), there exist \( i \) and \( j \) with \( 0 < i < j \leq m \) such that \( a_{i+1} + a_{i+2} + \ldots + a_j \) is divisible by \( m \).

Proof (in two cases):

- Consider \( m \) sums:
  \[ a_1, \ a_1 + a_2, \ a_1 + a_2 + a_3, \ \ldots, \ a_1 + a_2 + \ldots + a_m \]
- 1) If any of these sums is divisible by \( m \), then we are done!
- 2) Suppose not, each sum has a nonzero remainder when divided by \( m \).
- The possible remainders are \( 1, 2, 3, \ldots, m-1 \).
- By Pigeonhole principle, there are at least \( 2 \) sums with same remainder \( (r) \). We have:
  \[ a_1 + a_2 + \ldots + a_i = cm + r \]  \( \text{(i)} \)
  \[ a_1 + a_2 + \ldots + a_j = dm + r \]  \( \text{(ii)} \)
  where \( c, d \) and \( r \) are integers and assume \( i < j \)
- \( a_{i+1} + a_{i+2} + \ldots + a_j = (d-c) m \)  \( \text{(ii)} - (i) \)
  It is divisible by \( m \).