Overview

• HW#5 due in one week on 10/24
• MT2 in 2 weeks (10/31, relation/function)
• Last Lecture
  – Function: Definition: A special type of binary relation!!!
  – Function Properties: 1-1, Onto, 1-1 Correspondences
  – Function Operations: Inverse, Composition
• This Lecture completes lessons for Functions
  – Sum, Products
  – Sequence, Subsequence, Summation
  – Theory of Counting
  – Pigeonhole Principle
  – Generalized Pigeonhole Principle

Chapter 4. Functions
Cond.
Operations: Sum & Product of Functions

Let \( f_1 \) and \( f_2 \) be functions from \( A \) to \( \mathbb{R} \). Then the **sum** and the **product** of \( f_1 \) and \( f_2 \) are also functions from \( A \) to \( \mathbb{R} \) defined by:

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x)
\]

\[
(f_1 f_2)(x) = f_1(x) f_2(x)
\]

**Example:**

\( f_1(x) = 3x, \ f_2(x) = x + 5 \)

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5
\]

Let \( x = 10 \),

\[
(f_1 + f_2)(10) = 30 + 15 = 45
\]

\[
(f_1 f_2)(x) = f_1(x) f_2(x) = 3x (x + 5) = 3x^2 + 15x
\]

Let \( x = 10 \),

\[
(f_1 f_2)(10) = 300 + 150 = 450
\]

Sequence: list of objects

- **Definition:** A **sequence** lists up/order things in the set \( S \). But a **sequence** can also be defined as a function from a subset of integers (usually \( \{0,1,2,\ldots\} \) or \( \{1,2,3,\ldots\} \)) to a set \( S \). A set does not regard an order of elements by definition! But sequence does.

\[
\{0,\Delta,\star\} \rightarrow \{0,\Delta,\star\} \rightarrow \{(1,0),(2,\Delta),(3,\star)\}
\]

\[
\{0,\Delta,\star\} \rightarrow \{0,\Delta,\star\} \rightarrow \{(1,\Delta),(2,\star),(3,\star)\}
\]

\[
\{\Delta,\star,0\} \rightarrow \{(1,\Delta),(2,\star),(3,\star)\}
\]

\[
\{2,3\}
\]
Sequence: list of objects

• Definition: A sequence lists up/order things in the set S. But a sequence can also be defined as a function from a subset of integers (usually \{0,1,2…\} or \{1,2,3,…\}) to a set S. A set does not regard an order of elements by definition! But sequence does.

• Example 1: a simple sequence “a,b,c” is a sequence with 3 terms (finite sequence) “a,c,b”?

  \{(1,a),(2,b),(3,c)\} \quad \{(0,a),(1,c),(2,b)\}

• Example 2: a geometric progression is a sequence:
  
a, ar, ar^2, ar^3, … where a and r are real numbers i.e. \(f(n)=ar^n\), for \(n \in \mathbb{N}\) (infinite sequence)

• Definition: A subsequence is a shorter sequence with the same order as they do in the original sequence

• Example 3: Use sequence \(X = 1,2,3,5,8,13,21\)

  1,3,13,21 is a subsequence of X

  1,8,5 is not a subsequence of X

  1,2,3,5,8,13,21 is a subsequence of X
• Definition: Given a sequence \( a_m, a_{m+1}, \ldots, a_n \). We use the notation \( \sum_{i=m}^{n} a_i \) to represent the sum of \( a_m + a_{m+1} + \ldots + a_n \).

• Example: \( \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \)

Let \( x = 1 + 2 + 3 + \ldots + n \)
\[ x = \frac{n(n+1)}{2} \]

• Double Summations

Corresponding to a double loops in C or Java, there is also triple and quadruple summation corresponding to nested loops:

Example:
\[ \sum_{i=1}^{5} \left( \sum_{j=1}^{2} ij \right) = \sum_{i=1}^{5} (i + 2i) = \sum_{i=1}^{5} 3i = 3 + 6 + 9 + 12 + 15 = 45 \]
• Review: Given a function \( f: S \rightarrow T \)

The range \( R = f(S) = \{ f(s) \mid s \in S \} \subseteq T \).

Then \( |R| \leq |T| \) **note**: \(|X|\) is cardinality of \( X \)

If \( f \) is injection, 1-1, then \( |S| = |R| \leq |T| \)

For all \( a, b \) in \( S \), if \( f(a) = f(b) \), then \( a = b \)

If \( f \) is surjection, onto, then \( |S| \geq |R| = |T| \)

For all \( b \) in \( T \), we have \( a = f^{-1}(b) \) in \( S \)

If \( f \) is bijection, 1-1 correspondence, then \( |S| = |R| = |T| \)
• Review: Let $X$ be a set and $n \in \mathbb{N}$. If $|X| = |\{1,2,\ldots,n\}|$, then the **cardinality** of $X$ is $n$ and $X$ is **finite**. We say $X$ is **infinite** if $X$ is not finite.

• Definition: Let $X$ be a set. $X$ is **countably infinite** if $|X| = |\mathbb{N}| = \infty$.

• $X$ is **countable** if it is either finite or countably infinite. If a set is not countable, then it is **uncountable**.

• **Theorem:** If we can define a bijective function $f : \mathbb{N} \to X$, then $|\mathbb{N}| = |X|$ thus a set $X$ is countably infinite.

To show $X$ is countable

→ Find a function $f : \mathbb{N} \to X$, then prove that it is bijective,

→ List the elements of $X$ in some order.

• Example: $X = \{n \in \mathbb{N} \text{ and } n \text{ is even}\}$ is countably infinite.

  $f : \mathbb{N} \to X$ where $f(n) = 2n$

  $f$ is 1-1 correspondence (prove it!)

• Example: Integer set $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$ is countably infinite.

  Let us list $\mathbb{Z}$ as $0,-1,1,-2,2,-3,3,\ldots$

  then define $f : \mathbb{N} \to \mathbb{Z}$ where

  $f(0) = 0, f(2n) = n, f(2n-1) = -n$ for $n > 0$

  $f$ is 1-1 correspondence (prove it!)

• Example: $\mathbb{Q}^+ = \text{set of all positive rational numbers, } \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z}, n > 0\}$, is countably infinite.
• Find a way to list all members of \( \mathbb{Q}^+ \) as an infinite sequence and each real number is only listed once. Therefore, \( \mathbb{N} \to \mathbb{Q}^+ \) is 1-1 correspondence (bijection).

• Use **Cantor diagonalization** argument, list the positive rational numbers \( p/q \) with \( p+q=2 \), followed by those with \( p+q=3 \), then \( p+q=4 \), etc. If a number already listed, do not list them again.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \ldots \\
2 & \frac{1}{5} & \frac{2}{5} & \ldots & \frac{1}{9} & \frac{2}{9} & \frac{3}{9} & \ldots \\
3 & \frac{1}{10} & \frac{2}{10} & \ldots & \frac{1}{17} & \frac{2}{17} & \frac{3}{17} & \ldots \\
4 & \frac{1}{18} & \frac{2}{18} & \ldots & \frac{1}{25} & \frac{2}{25} & \frac{3}{25} & \ldots \\
5 & & & & & & \ldots \\
6 & & & & & & \ldots \\
7 & & & & & & \ldots \\
8 & & & & & & \ldots \\
\vdots & & & & & & \ddots \\
\end{array}
\]

The first few terms are 1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, 5…

- Example: \( \mathbb{Q} \) set of all **rational numbers** is countably infinite. List all positive and negative rational numbers using previous examples for \( \mathbb{Z} \) and \( \mathbb{Q}^+ \).

- Example: Real numbers set \( \mathbb{R} \) is **uncountable**. We will show that the subset \( X \) of all real numbers \( \mathbb{R} \) only between 0 and 1 is uncountable. (If a subset of \( A \) is uncountable, then \( A \) must be uncountable!!)

Suppose that \( X \) is countable and then arrive a contradiction. Suppose \( X=\{x| x \in \mathbb{R} \text{ and } 0<x<1\} \) is countable, then we can list all real numbers between 0<x<1. Let \( r_i \in X \) for all \( i=1,2,3,\ldots \)

- \( r_1 = 0. \, d_{11} \, d_{12} \, d_{13} \ldots \)
- \( r_2 = 0. \, d_{21} \, d_{22} \, d_{23} \ldots \)
- \( r_3 = 0. \, d_{31} \, d_{32} \, d_{33} \ldots \)
- \( \ldots \)
- \( r_i = 0. \, d_{i1} \, d_{i2} \, d_{i3} \ldots \)

\[\text{note: } d_{xy} \in \{0,1,2,3,4,5,6,7,8,9\}\]
Now, let’s construct a new real number \( r^* \) s.t. \( 0 < r^* < 1 \) as follows:
\[
r^* = 0. \ d_1 \ d_2 \ d_3 \ldots \ d_i \ldots
\]
for all \( i = 1, 2, 3, \ldots \) such that if \( d_{ii} \neq 4 \), then \( d_i = 4 \); otherwise \( d_i = 5 \).

Example:
- \( r_1 = 0. \ 2 \ 3 \ 7 \ 9 \ 4 \ 1 \ldots \)
- \( r_2 = 0. \ 4 \ 4 \ 5 \ 9 \ 0 \ 3 \ldots \)
- \( r_3 = 0. \ 0 \ 9 \ 1 \ 1 \ 8 \ 7 \ldots \)
- \( r_4 = 0. \ 9 \ 5 \ 6 \ 4 \ 7 \ 9 \ldots \)
- \( r_5 = 0. \ 2 \ 2 \ 4 \ 5 \ 6 \ 1 \ldots \)
- \( \ldots \)
- \( r^* = 0. \ 4 \ 5 \ 4 \ 5 \ 4 \ldots \)

Clearly, \( r^* \notin X \) since for each \( r_i \in X \), \( d_i \neq d_{ii} \), however we supposed \( 0 < r^* < 1 \). So this is a contradiction.

Therefore, all the real numbers between 0 and 1 cannot be listed thus uncountable. Then \( R \) is uncountable because \( R \) contains more elements than \( X \).

**Pigeonhole Principle:** If \( x \) items are placed into \( y \) bins where \( x > y \), then there is one bin which contains at least two items.

Note: text book specified this property by using function notation

**Proof by contradiction:** Suppose \( x (> y) \) items are placed into \( y \) bins and suppose no bin with two or more items. Each bin has 0 or 1 item. The maximum total number of items in \( y \) bins is then \( y \), but \( y < x \). This contradiction proves the result.

Example: How many people must be in a room to guarantee that two people have last names that begin with the same letter?

There are 26 letters(or bins). If there are 27 people, then at least 2 people will have last names beginning with the same letter.
Example: The population of city x is about 40,000. If each resident has three initials, is it true that there must be at least 2 individuals with the same initials?

How many possible combination of 3 letters?
26 * 26 * 26 = 17,576 < 40,000

By Pigeonhole principle, there must be at least 2 individual with the same initials

Generalized Pigeonhole Principle: If x items are placed into y bins where x > y, then there is one bin which contains at least ⌈x/y⌉ (note: ceiling) items.

Example: For above example, it is true that there must be at least 3 individual with the same initials, i.e ⌈40000/17576⌉ = 3.

Example: What is the minimum number of people in a group there must be so that there are at least 3 who were born in the same month?

There are 12 months (bins). With > 12 people, at least 2 people who were born in the same month. With > 24 people, at least 3 people who were born in the same month. So, minimum number is 25 people.

Example: Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

Ans: ??
Application - Problem with divisors:
Let $m \in \mathbb{N}$. Given $m$ integers $a_1, a_2, \ldots, a_m$, there exist $i$ and $j$ with $0 < i < j \leq m$ such that $a_{i+1} + a_{i+2} + \ldots + a_j$ is divisible by $m$.

Proof (in two cases):
- Consider $m$ sums:
  
  $a_1$, $a_1 + a_2$, $a_1 + a_2 + a_3$, \ldots, $a_1 + a_2 + \ldots + a_m$

- 1) If any of these sums is divisible by $m$, then we are done!

- 2) Suppose not, each sum has a nonzero remainder when divided by $m$.

- The possible remainders are 1, 2, 3,\ldots, $m-1$.

- By Pigeonhole principle, there are at least 2 sums with same remainder ($r$). We have:

  $a_1 + a_2 + \ldots + a_i = cm + r$ \hspace{1cm} (i)

  $a_1 + a_2 + \ldots + a_j = dm + r$ \hspace{1cm} (ii)

  where $c$, $d$ and $r$ are integers and assume $i < j$

- $a_{i+1} + a_{i+1} + \ldots + a_j = (d-c) m$ \hspace{1cm} (ii) – (i)

  It is divisible by $m$. 