• Stay well! Spr break (3/24,26) + Cesar Chavez Day (3/31)
• Next Lec on 4/2: An essay homework over the break
• MT2 in 3 weeks on 4/9: Zoom, Things upto Functions, 2 page cheat sheets, PDF upload, Start to study for it!
• HW#5 due on 4/7. Questions on iLearn now. Work on it!
• Last Lecture
  – Function: Definition: A special type of binary relation!!
  – Function Properties: 1-1, Onto, 1-1 Correspondences
  – Function Operations: Inverse, Composition.
• This Lecture: Completes Functions
  – Function Operations: Sum, Product
  – Sequence, Subsequence, Summation
  – Theory of Counting
  – Pigeonhole Principle after the break
  – Generalized Pigeonhole Principle

Chapter 4. Functions
Cond.
Operations: Sum & Product of Functions

Let \( f_1 \) and \( f_2 \) be functions from \( A \) to \( \mathbb{R} \). Then the sum and the product of \( f_1 \) and \( f_2 \) are also functions from \( A \) to \( \mathbb{R} \) defined by:

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x)
\]

\[
(f_1 f_2)(x) = f_1(x) f_2(x)
\]

Example:

\[ f_1(x) = 3x, \quad f_2(x) = x + 5 \]

\((f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5\)

Let \( x = 10 \), \( f_1(10) + f_2(10)=30+15=45 \) \( (f_1 + f_2)(10) =40+5=45 \)

\((f_1 f_2)(x) = f_1(x) f_2(x) = 3x (x + 5) = 3x^2 + 15x\)

Let \( x = 10 \), \( f_1(10) * f_2(10)=30*15=450 \)

\((f_1 f_2)(10) =300+150=450\)

Sequence: list of objects

- Definition: A sequence is a function from a subset of integers (usually \( \{0,1,2,\ldots\} \) or \( \{1,2,3,\ldots\} \)) to a set \( S \). A sequence lists up/order things in the set \( S \). A set does not regard an order of elements by definition! But sequence does.

\[
\text{sequence} = \{(1, \text{cat}), (2, \text{dog}), (3, \text{pig})\}
\]

Making a sequence:

1. Is to pick elements in \( S \) in an order placing them from left to right!
2. Interpreted it as a function
3. \( \text{location index!} \)
4. \( \text{order doesn't matter!} \)

\[
\text{sequence} = \{ \text{cat, dog, pig} \}
\]
Sequence: list of objects

• Definition: A sequence is a function from a subset of integers (usually \{0,1,2\ldots\} or \{1,2,3,\ldots\}) to a set \(S\). A sequence lists up/order things in the set \(S\). A set does not regard an order of elements by definition! But sequence does.

• Example 1: a simple sequence “\(a,b,c\)” is a sequence with 3 terms (finite sequence) “\(a,c,b\)”?
\[
\{(1,a),(2,b),(3,c)\} \rightarrow f: \mathbb{N} \rightarrow S
\]

• Example 2: a geometric progression is a sequence: 
\(a, ar, ar^2, ar^3, \ldots\) where \(a\) and \(r\) are real numbers
i.e. \(f(n)=ar^n\), for \(n \in \mathbb{N}\) (infinite sequence)

• Definition: A subsequence is a shorter or the same length sequence with the same order as they do in the original sequence

• Example 3: Use sequence \(X = 1,2,3,5,8,13,21\)

\[
X \equiv 1,3,13,21
\]

\[
\{1,8\}
\]

\[
X = 1,2,3,5,8,13,21
\] is a subsequence of \(X\)

\[
1,8,5
\] is not a subsequence of \(X\)

\[
X \equiv 1,2,3,5,8,13,21\] is a subsequence of \(X\)
• Definition: Given a sequence $a_m, a_{m+1}, ..., a_n$. We use the notation $\sum_{j=m}^{n} a_j$ to represent summation $a_m + a_{m+1} + ... + a_n$.

• Example: $\sum_{j=1}^{n} j = 1+2+3+...+n = n(n+1)/2$

  Let $x = 1+2+3+...+n$
  $\therefore x = n(n-1)+...+1$
  $2x = n(n+1)$
  $x = n(n+1)/2$

• Example: $\sum_{k=0}^{n} ar^k, \forall r \neq 0 = \frac{ar^{n+1} - a}{r-1}, r \neq 1$

• Let $x = a + ar + ar^2 + ... + ar^n$
  $\therefore x(r-1) = (ar^{n+1} - a)$
  $x(r-1) = (ar^{n+1} - a)$

Double Summations

Corresponding to a double loops in C or Java, there is also triple and quadruple summation corresponding to nested loops:

Example:
\[
\sum_{i=1}^{5} \left( \sum_{j=1}^{2} ij \right) = \sum_{i=1}^{5} (i + 2i) = \sum_{i=1}^{5} 3i
\]

Triple Summation:
\[
\sum_{x=1}^{5} \sum_{y=1}^{2} f(x, y, z) = f(1, 1, 1) + f(1, 1, 2) + f(1, 2, 1) + f(1, 2, 2) + f(2, 1, 1) + f(2, 1, 2) + f(2, 2, 1) + f(2, 2, 2) + f(3, 1, 1) + f(3, 1, 2) + f(3, 2, 1) + f(3, 2, 2) + f(4, 1, 1) + f(4, 1, 2) + f(4, 2, 1) + f(4, 2, 2) + f(5, 1, 1) + f(5, 1, 2) + f(5, 2, 1) + f(5, 2, 2) + f(5, 3, 1) + f(5, 3, 2) + f(5, 3, 3)
\]
Let's think about the size of sets.
When you have an infinite set $A$
$|A| = \infty$!

**Theory of Counting**

A $\rightarrow |A| = \infty$
B: add 1 new element to A
C: add 100 new elements to A

Which one is longer?

$|A| < |B| < |C|$?

No

$|A| = |B| = |C| = \infty \ldots$

• Review: Given a function $f: S \rightarrow T$
The range $R = f(S) = \{ f(s) \mid s \in S \} \subseteq T$.
Then $|R| \leq |T|$ note: $|X|$ is cardinality of $X$

If $f$ is injection, 1-1, then $|S| = |R| \leq |T|$

For all $a, b$ in $S$, if $f(a) = f(b)$, then $a = b$

If $f$ is surjection, onto, then $|S| \geq |R| = |T|$

For all $b$ in $T$, we have $a = f^{-1}(b)$ in $S$

If $f$ is bijection, 1-1 correspondence, then $|S| = |R| = |T|$

$|S| = |R| \leq |T|$ and $|S| \geq |R| = |T|$ at the same time.
• Review: Let X be a set and \( n \in \mathbb{N} \). If \( |X| = |\{0,1,2,\ldots,n-1\}| \), then the **cardinality** of X is \( n \) and X is **finite**. We say X is **infinite** if X is not finite.

• Definition: Let X be a set. X is **countably infinite** if \( |X| = |\mathbb{N}| = \infty \).

• X is **countable** if it is either finite or countably infinite. If a set is not countable, then it is **uncountable**.

• **Theorem:** If we can define a bijective function \( f : \mathbb{N} \to X \), then \( |\mathbb{N}| = |X| \) thus a set X is countably infinite.

To show X is countable:
1. Find a function \( f : \mathbb{N} \to X \), then prove that it is bijective,
2. List the elements of X in some order.

• Example: \( X = \{ n \in \mathbb{N} \) and n is even\} is countably infinite.

\[ f : \mathbb{N} \to X \text{ where } f(n) = 2n \]

f is 1-1 correspondence (prove it!)

• Example: Integer set \( \mathbb{Z} = \{ \ldots,-2,-1,0,1,2,\ldots \} \) is countably infinite.

Let us list \( \mathbb{Z} \) as \( 0, -1, 1, -2, 2, -3, 3, \ldots \)

then define \( f : \mathbb{N} \to \mathbb{Z} \) where

\[ f(0)=0, f(2n)=n, f(2n-1)=-n \text{ for } n>0 \]

f is 1-1 correspondence (prove it!)

• Example: \( \mathbb{Q}^+ = \text{set of all positive rational numbers} \), \( \{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z}, n > 0 \} \), is countably infinite.
• Find a way to list all members of \( \mathbb{Q}^+ \) as an infinite sequence and each real number is only listed once. Therefore, \( \mathbb{N} \to \mathbb{Q}^+ \) is 1-1 correspondence (bijection).

• Use following argument, list the positive rational numbers \( p/q \) with \( p+q=2 \), followed by those with \( p+q=3 \), then \( p+q=4 \), etc. If a number already listed, do not list them again.

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The first few terms are 1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, 5...

• Example: \( \mathbb{Q} \) = set of all rational numbers is countably infinite.

List all positive and negative rational numbers using previous examples for \( \mathbb{Z} \) and \( \mathbb{Q}^+ \).

• Example: Real numbers set \( \mathbb{R} \) is uncountable. We only need to show that the subset \( X \) of all real numbers \( \mathbb{R} \) between 0 and 1 is uncountable. (If a subset of \( \mathbb{R} \) is uncountable, then \( \mathbb{R} \) must be uncountable!!!) \( \leftarrow \) Cantor diagonalization \( X \subset \mathbb{R} \Rightarrow |X| \leq |\mathbb{R}| \)

Suppose that \( X \) is countable and then arrive a contradiction.

Suppose \( X = \{x | x \in \mathbb{R} \text{ and } 0 < x < 1 \} \) is countable, then we can list all real numbers between 0<x<1. Let \( r_i \in X \) for all \( i = 1, 2, 3, ... \)

\[
\begin{align*}
  r_1 &= 0. d_{11} d_{12} d_{13} \ldots \\
  r_2 &= 0. d_{21} d_{22} d_{23} \ldots \\
  r_3 &= 0. d_{31} d_{32} d_{33} \ldots \\
  \vdots \\
  r_i &= 0. d_{i1} d_{i2} d_{i3} \ldots \quad \text{(digits whose digit index \& number index one the same in this list.)}
\end{align*}
\]

\[
\begin{align*}
  d_{xy} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}
\end{align*}
\]
Now, let’s construct a new real number $r^*$ s.t. $0 < r^* < 1$ as follows:

$$r^* = 0. \, d_1 \, d_2 \, d_3 \ldots d_i \ldots$$

for all $i=1,2,3,\ldots$ such that if $d_{ii} \neq 4$, then $d_i=4$; otherwise $d_i=5$

Clearly, $r^* \notin X$ since for each $r_i \in X$, $d_i \neq d_{ii}$, however we supposed $0 < r^* < 1$. So this is a contradiction.

Therefore, all the real numbers between 0 and 1 cannot be listed thus uncountable. Then $R$ is uncountable because $R$ contains more elements than $X$.

Example:

- $r_1 = 0.237941\ldots$
- $r_2 = 0.45903\ldots$
- $r_3 = 0.09187\ldots$
- $r_4 = 0.95679\ldots$
- $r_5 = 0.24561\ldots$
- $\ldots$
- $r^* = 0.45454\ldots$

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**Pigeonhole Principle**: If $x$ items are placed into $y$ bins where $x > y$, then there is one bin which contains at least two items.

Note: textbook specified this property by using function notation

**Proof by contradiction**: Suppose $x > y$ items are placed into $y$ bins and suppose no bin with two or more items. Each bin has 0 or 1 item. The maximum total number of items in $y$ bins is then $y$, but $y < x$. This contradiction proves the result

Example: How many people must be in a room to guarantee that two people have last names that begin with the same letter?

There are 26 letters(or bins). If there are 27 people, then at least 2 people will have last names beginning with the same letter.
Example: The population of city x is about 40,000. If each resident has three initials, is it true that there must be at least 2 individuals with the same initials?

How many possible combination of 3 letters?
$26 \times 26 \times 26 = 17,576 < 40,000$

By Pigeonhole principle, there must be at least 2 individual with the same initials

**Generalized Pigeonhole Principle:** If x items are placed into y bins where x > y, then there is one bin which contains at least $\lceil x/y \rceil$ (note: ceiling) items.

Example: For above example, it is true that there must be at least 3 individual with the same initials, i.e $\lceil 40000/17576 \rceil = 3$.

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Example: What is the minimum number of people in a group there must be so that there are at least 3 who were born in the same month?

There are 12 months (bins). With > 12 people, at least 2 people who were born in the same month. With > 24 people, at least 3 people who were born in the same month. So, minimum number is 25 people.

Example: Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

Ans: ??
Application - Problem with divisors:
Let $m \in \mathbb{N}$. Given $m$ integers $a_1, a_2, \ldots, a_m$, there exist $i$ and $j$ with $0 < i < j \leq m$ such that $a_{i+1} + a_{i+2} + \ldots + a_j$ is divisible by $m$.

Proof (in two cases):
• Consider $m$ sums:
  $a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \ldots + a_m$
• 1) If any of these sums is divisible by $m$, then we are done!
• 2) Suppose not, each sum has a nonzero remainder when divided by $m$.
• The possible remainders are $1, 2, 3, \ldots, m-1$.
• By Pigeonhole principle, there are at least 2 sums with same remainder ($r$). We have:
  $a_1 + a_2 + \ldots + a_i = cm + r$ \hspace{1cm} (i)
  $a_1 + a_2 + \ldots + a_j = dm + r$ \hspace{1cm} (ii)
  where $c$, $d$ and $r$ are integers and assume $i < j$
• $a_{i+1} + a_{i+1} + \ldots + a_j = (d-c) m$ \hspace{1cm} (ii) – (i)
It is divisible by $m$. 