Overview

- HW4 Due & HW5 assignment will be online!
- HW5 is due in 1.5 weeks on Oct 21, Thursday
- Last Lecture: Completed Relation
  - Equivalence Class Cond
  - Partition
  - Partial Ordering
  - Relations with multiple sets
  - (Relational Database)

- This Lecture: Function
  - Function Definition
  - Function Properties: Injective, Surjective, Bijective
  - Function Operations: Composition

Chapter 4. Functions
Functions = special cases of binary relations from a set $S$ to a set $T$.

- Given sets $S$ and $T$, a function (mapping) $f$ from $S$ to $T$, noted $f: S \rightarrow T$, is a subset of $S \times T$ where each member of $S$ appears exactly once as the first component of an ordered pair.

- For $A \subseteq S$, $f(A)$ denotes $\{f(a) | a \in A\}$.

Examples: Which of the following are functions from the domain and codomain indicated, or not? For those that are not, why not?

$f: S \rightarrow T$ where $S = T = \{1,2,3\}, f = \{(1,1),(2,3),(3,1),(2,1)\}$

Ans: Not a function; $2 \in S$ has two values associated with it.

$g: \mathbb{Z} \rightarrow \mathbb{N}$ where $g$ is defined by $g(x) = |x|$ (the absolute value of $x$)

Ans: Yes, it is a function

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More examples:

\[ f: S \rightarrow T \text{ where } S \text{ is the set of all people in SF,} \]
\[ T \text{ is the set of all driver license numbers, and } f \text{ associates} \]
\[ \text{with each person with that person’s driver license number} \]

Ans: Not a function in our definition; not every member

of \( S \) has a driver license number

(Partial Function vs. Total Function: \( y = 1/x \))

g: \( \mathbb{R} \rightarrow \mathbb{R} \) and \( g \) is defined by the graph as follows:

Ans: Yes

In summary, a complete definition of a function requires
giving three things:

• Its domain
• Its codomain
• The association
  • A collection of ordered pairs
  • An equation
  • Verbal description
  • A graph
Some function $f$ can also be defined \textbf{recursively}.

Example:

Given $f: \mathbb{N} \rightarrow \mathbb{N}$ where $f(n) = 3^n$

Recursive definition for $f$ is

$f(0) = 1$, $f(n) = 3*f(n-1)$ for $n>0$

$f(1) = 3*f(0) = 3*1 = 3^1$

$f(2) = 3*f(1) = 3*3 = 9 = 3^2$

$f(3) = 3*f(2) = 3*9 = 27 = 3^3$

\ldots

$f(n) = 3^n$

(can be proved by mathematical induction!)

\textbf{Definition}: Two functions, $f, g: X \rightarrow Y$, are \textbf{equal} if and only if they contain the \textbf{same domain}, \textbf{same co-domain} and \textbf{same ordered pairs}

\begin{itemize}
  \item \textbf{e.g.} Let $S = \{1, 2, 3\}$ and $T = \{1, 4, 9\}$.
  \item The function $f: S \rightarrow T$ is denoted by $f = \{(1, 1), (2, 4), (3, 9)\}$.
  \item The function $g: S \rightarrow T$ is denoted by the equation $g(n) = \sum_{k=1}^{n} (4k - 2) / 2$
\end{itemize}

It is true that $f = g$

\textbf{Note}: can be checked easily with all elements of $S$

Also, $\sum$ is a notation for \textbf{summation}, will define later.
**Definition:** Let $f: S \rightarrow T$ be an arbitrary function with domain $S$ and codomain $T$. Part of the definition of a function is that every member of $S$ has an image under $f$ and that all the images are members of $T$; the set $R$ of all such images is called the **range** of the function $f$.

Thus $R = \{ f(s) \mid s \in S \}$ or $R = f(S)$ and clearly $R \subseteq T$

Example: Let us take a look at the function $f: S \rightarrow T$ with

$S = \{ \text{Linda, Max, Kathy, Peter} \}$
$T = \{ \text{Boston, New York, Hong Kong, Moscow} \}$

Let us specify $f$ as follows:

$f(\text{Linda}) = \text{Moscow} \quad f(\text{Max}) = \text{Boston}$
$f(\text{Kathy}) = \text{Hong Kong} \quad f(\text{Peter}) = \text{Boston}$

$R = \{ \text{Moscow, Boston, Hong Kong} \} \subseteq T$

**Properties of Functions**

1. **One-to-One (1-1, Injective)**

**Definition:** A function $f: S \rightarrow T$ is **One-to-One**, or **injective**, if no member of $T$ is the image of two or more distinct elements of $S$ under $f$.

**Note:** To prove that function $f$ is injective, show that, for all $s_1$ and $s_2$ in $S$, $f(s_1) = f(s_2) \rightarrow s_1 = s_2$.

Example: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = 2x$

$f$ is one-to-one. For any $a, b \in \mathbb{R}$ (domain), if $f(a) = f(b)$ means $2a = 2b$. It follows that $a = b$. □

Example: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = x^2$

$f$ is not one-to-one. **Counterexample:** $f(-2) = f(2) = 4$ □
2. Onto (Surjective)

- **Definition:** A function \( f: S \rightarrow T \) is an onto, or surjective, function if the range \( R = f(S) \) of \( f \) equals the codomain \( T \) of \( f \),

\[ R = T \]

**Note.** To prove that a function is surjective, **show that, for any \( b \) in \( T \), there exists an element \( a \) in \( S \) that satisfies \( f(a) = b \)** (Every member of codomain \( T \) is an image/not missed).

- Example: Let \( f: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) where \( f(x) = x^2 \)

\( \mathbb{R}^+ \) means \( x \geq 0 \) and \( x \in \mathbb{R} \)

\( f \) is onto. For any \( b \) in codomain \( \mathbb{R}^+ \), we have \( b = f(a) = a^2 \).

Solving this about “\( a \)” leads to \( a = \pm \sqrt{b} \). Since \( a \in \mathbb{R}^+ \), it must be that \( a = \sqrt{b} \), which shows that for any \( b \in T = \mathbb{R}^+ \) we have corresponding \( a \in S = \mathbb{R}^+ \). \( \square \)

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3. One-to-one correspondence (1-1 correspondences, Bijective)

**Definition:** A function \( f: S \rightarrow T \) is **bijective** (or one-to-one correspondence) if it is both one-to-one and onto

**Note :** To prove that a function \( f \) is a bijection requires proving that \( f \) is both onto and one-to-one.

Example: Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) where \( f(x) = x^3 \) is bijective

\( \mathbb{R} \)

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**Composition of functions:**

**Definition:** Let \( f: S \to T \) and \( g: T \to U \). Then the composition function, \( g \circ f \), is a function from \( S \) to \( U \) defined by \((g \circ f)(s) = g(f(s))\).

**Note1:** this is same as the composition of relations

**Note2:** the function \( g \circ f \) is applied from right to left; function \( f \) is applied first and then function \( g \).

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**e.g.** Let \( f: \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^2 \).

Let \( g: \mathbb{R} \to \mathbb{R} \) be defined by \( g(x) = \lfloor x \rfloor \).

Note: the floor function \( \lfloor x \rfloor \) associates with each real number \( x \) the greatest integer less than or equal to \( x \) (see previous examples).

What is the value of \((g \circ f)(2.5)\)?

\[
(g \circ f)(2.5) = g(f(2.5)) = g((2.5)^2) = g(6.25) = \lfloor 6.25 \rfloor = 6
\]

What is the value of \((f \circ g)(2.5)\)?

\[
(f \circ g)(2.5) = f(g(2.5)) = f(\lfloor 2.5 \rfloor) = f(2) = 2^2 = 4
\]
Theorem: Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be functions

- If $f$ and $g$ are both 1-1, then $g \circ f$ is 1-1
- If $f$ and $g$ are both onto, then $g \circ f$ is onto
- If $f$ and $g$ are both 1-1 correspondences, then $g \circ f$ is a 1-1 correspondence
- If $g \circ f$ is 1-1, then $f$ is 1-1
- If $g \circ f$ is onto, then $g$ is onto