Overview

- HW#4 due now & Take a HW#5 handout & old HWs
- MT2 will be on 10/31 covering materials: relation and function. Start to prepare for it.
- Last Lecture: completed materials on Relation
  - Equivalence Class Cond
  - Partition
  - Partial Ordering
  - Relations with multiple sets
  - (Relational Database) ← recommended reader. Not in MT2.
- This Lecture: Function
  - Function Definition
  - Function Properties: Injective, Surjective, Bijective
  - Function Operations: Composition, Inverse
  - Sequence
  - Summation

Chapter 4, Functions
Definition

Functions = special cases of binary relations from a set $S$ to a set $T$.

• Given sets $S$ and $T$, a function $f$ from $S$ to $T$, denoted $f: S \rightarrow T$, is a subset of $S \times T$ where each member of $S$ appears exactly once as the first component of an ordered pair.

• For $A \subseteq S$, $f(A)$ denotes $\{f(a) | a \in A\}$.

Examples: Which of the following are functions from the domain and codomain indicated, or not? For those that are not, why not?

$f: S \rightarrow T$ where $S = T = \{1, 2, 3\}, f = \{(1, 1), (2, 3), (3, 1), (2, 1)\}$

Ans: Not a function; 2 $\in$ $S$ has two values associated with it

g: $\mathbb{Z} \rightarrow \mathbb{N}$ where $g$ is defined by $g(x) = |x|$ (the absolute value of $x$)

Ans: Yes, it is a function
More examples:

\( f: S \rightarrow T \) where \( S \) is the set of all people in SF, 
\( T \) is the set of all driver license numbers, and \( f \) associates 
with each person with that person’s driver license number

Ans: Not a function in our definition; not every member 
of \( S \) has a driver license number 
(Partial Function vs. Total Function: \( y = 1/x \))

\( g: \mathbb{R} \rightarrow \mathbb{R} \) and \( g \) is defined by the graph as follows:

Ans: Yes

In summary, a complete definition of a function requires 
giving:

- Its domain
- Its codomain
- The association
  - A collection of ordered pairs
  - An equation
  - Verbal description
  - A graph
Some function $f$ can also be defined **recursively**.

**Example:**

Given $f: \mathbb{N} \rightarrow \mathbb{N}$ where $f(n) = 3^n$

Recursive definition for $f$ is

- $f(0) = 1$
- $f(n) = 3f(n-1)$ for $n > 0$

$f(1) = 3f(0) = 3 \times 1 = 3^1$

$f(2) = 3f(1) = 3 \times 3 = 9 = 3^2$

$f(3) = 3f(2) = 3 \times 9 = 27 = 3^3$

....

$f(n) = 3^n$

(can be proved by mathematical induction!)

**Definition:** Two functions, $f, g: X \rightarrow Y$, are **equal** if and only if they contain the **same domain**, **same co-domain** and **same ordered pairs**

e.g. Let $S = \{1, 2, 3\}$ and $T = \{1, 4, 9\}$.

The function $f: S \rightarrow T$ is denoted by $f = \{(1, 1), (2, 4), (3, 9)\}$.

The function $g: S \rightarrow T$ is denoted by the equation

$$
\sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} (4k - 2) \quad g(n) = \frac{\sum_{k=1}^{n} (4k - 2)}{2}
$$

It is true that $f = g$

Note: can be checked easily with all elements of $S$.

Also, $\sum$ is a notation for **summation**, will define later.
Definition: Let $f: S \rightarrow T$ be an arbitrary function with domain $S$ and codomain $T$. Part of the definition of a function is that every member of $S$ has an image under $f$ and that all the images are members of $T$; the set $R$ of all such images is called the **range** of the function $f$.

Thus $R = \{ f(s) \mid s \in S \}$ or $R = f(S)$ and **clearly** $R \subseteq T$

Example: Let us take a look at the function $f: S \rightarrow T$ with

$S = \{\text{Linda}, \text{Max}, \text{Kathy}, \text{Peter}\}$

$T = \{\text{Boston, New York, Hong Kong, Moscow}\}$

Let us specify $f$ as follows:

$f(\text{Linda}) = \text{Moscow}$  \quad \quad  f(\text{Max}) = \text{Boston}$

$f(\text{Kathy}) = \text{Hong Kong}$  \quad \quad  f(\text{Peter}) = \text{Boston}$

$R = \{\text{Moscow, Boston, Hong Kong}\} \subseteq T$

Properties of Functions

1. One-to-One (Injective)

**Definition:** A function $f: S \rightarrow T$ is **One-to-One**, or **injective**, if no member of $T$ is the image of two or more distinct elements of $S$ under $f$.

$$|S| = |R| \leq |T|$$

**Note:** To prove that function $f$ is injective, show that, for all $s_1$ and $s_2$ in $S$, $f(s_1) = f(s_2) \rightarrow s_1 = s_2$.

Example: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ where $f(x)=2x$

$f$ is one-to-one. For any $a, b \in \mathbb{R}$, if $f(a) = f(b)$ means $2a = 2b$.

It follows that $a = b$.

Example: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ where $f(x)=x^2$

$f$ is not one-to-one. **Counterexample**: $f(-2) = f(2) = 4$
### 2. Onto (Surjective)

**Definition:** A function \( f: S \rightarrow T \) is an **onto**, or **surjective**, function if the range \( R = f(S) \) of \( f \) equals the codomain \( T \) of \( f \),

\[
|S| \geq |T| = |R| \quad \Rightarrow \quad R = T
\]

**Note.** To prove that a function is surjective, show that, for any \( b \) in \( T \), there exists an element \( a \) in \( S \) that satisfies \( f(a) = b \) (Every member of codomain \( T \) is an image/not missed).

- **Example:** Let \( f: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+} \) where \( f(x) = x^2 \)
  \( \mathbb{R}^{+} \) means \( x \geq 0 \) and \( x \in \mathbb{R} \)
  \( f \) is onto. For any \( b \) in codomain \( \mathbb{R}^{+} \), we have \( b = f(a) = a^2 \).
  Solving this about “\( a \)” leads to \( a = \pm \sqrt{b} \). Since \( a \in \mathbb{R}^{+} \), it must be that \( a = \sqrt{b} \), which shows that for any \( b \in T = \mathbb{R}^{+} \) we have corresponding \( a \in S = \mathbb{R}^{+} \).

### 3. One-to-one correspondence (Bijective)

**Definition:** A function \( f: S \rightarrow T \) is **bijective** (or **one-to-one correspondence**) if it is both **one-to-one** and **onto**

**Note :** To prove that a function \( f \) is a bijection requires proving that \( f \) is both onto and one-to-one.

**Example:** Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) where \( f(x) = x^3 \) is bijective

\[
\text{If } f(a) = f(b) \text{ then } a^3 = b^3 \text{ } \text{ } \text{ } \text{ } \Rightarrow \text{ } \text{ } \text{ } \text{ } a = b
\]

\[
\text{For any } b \in \mathbb{R} \text{ } \text{ } \text{ } \text{ } \Rightarrow b = a^3 \text{ } \text{ } \text{ } \text{ } \Rightarrow a = \sqrt[3]{b} \text{ } \text{ } \text{ } \text{ } \Rightarrow a \in \mathbb{R}
\]
Composition of functions:

Definition: Let \( f: S \rightarrow T \) and \( g: T \rightarrow U \). Then the composition function, \( g \circ f \), is a function from \( S \) to \( U \) defined by \((g \circ f)(s) = g(f(s))\).

Note 1: this is same as the composition of relations

Note 2: the function \( g \circ f \) is applied from right to left; function \( f \) is applied first and then function \( g \).

e.g. Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = x^2 \).

Let \( g: \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( g(x) = \lfloor x \rfloor \).

Note: the floor function \( \lfloor x \rfloor \) associates with each real number \( x \) the greatest integer less than or equal to \( x \) (see previous examples).

What is the value of \((g \circ f)(2.5)\)?

\[(g \circ f)(2.5) = g(f(2.5)) = g((2.5)^2) = g(6.25) = \lfloor 6.25 \rfloor = 6\]

What is the value of \((f \circ g)(2.5)\)?

\[(f \circ g)(2.5) = f(g(2.5)) = f(\lfloor 2.5 \rfloor) = f(2) = 2^2 = 4\]
Theorem: Let $f: X \to Y$ and $g: Y \to Z$ be functions

- If $f$ and $g$ are both 1-1, then $g \circ f$ is 1-1
- If $f$ and $g$ are both onto, then $g \circ f$ is onto
- If $f$ and $g$ are both 1-1 correspondences, then $g \circ f$ is a 1-1 correspondence
- If $g \circ f$ is 1-1, then $f$ is 1-1
- If $g \circ f$ is onto, then $g$ is onto

Operations: Inverse of Functions & Identity Function

Definition: Let $f = \{(x,y) \in A \times B : f(x) = y\}$. The inverse of $f$, denoted by $f^{-1}$,

$$f^{-1} = \{(y,x) \in B \times A : f(y) = x\}.$$ 

Theorem: Let $f: S \to T$. Then $f$ is a bijection (1-1 correspondence) if and only if $f^{-1}$ exists.

Definition: The function that maps each element of a set $S$ to itself is called the identity function on $S$, i.e. $i(x) = x$.

Theorem: Composition of a bijective function and its inverse:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$$

i.e. The composition of a function and its inverse is the identity function $i(x) = x$. 
**Operations: Sum & Product of Functions**

Let \( f_1 \) and \( f_2 \) be functions from \( A \) to \( \mathbb{R} \). Then the **sum** and the **product** of \( f_1 \) and \( f_2 \) are also functions from \( A \) to \( \mathbb{R} \) defined by:

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x)
\]

\[
(f_1 f_2)(x) = f_1(x) f_2(x)
\]

**Example:**

\( f_1(x) = 3x \), \( f_2(x) = x + 5 \)

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5
\]

Let \( x = 10 \), \( f_1(10) + f_2(10) = 30 + 15 = 45 \)

\[
(f_1 + f_2)(10) = 40 + 5 = 45
\]

\[
(f_1 f_2)(x) = f_1(x) f_2(x) = 3x (x + 5) = 3x^2 + 15x
\]

Let \( x = 10 \), \( f_1(10) f_2(10) = 30 \times 15 = 450 \)

\[
(f_1 f_2)(10) = 300 + 150 = 450
\]