Overview

• HW#4 due next Tuesday. Work on it!

• Last Lecture
  – Binary Relations: Set of pairs, A subset of SxS
  – Operations: Inverse, Compositions
  – Properties: Identity, Empty, Universal, Reflexive,
  – Types: Irreflexive, Symmetric, Antisymmetric, Transitive

• This Lecture
  – Exercise on relation types
  – Nth power of R
  – Closure
  – Equivalence Relations, Equivalence Class
  – Partition
  – Partial Ordering
  – Relations with multiple sets
  – Relational Database

• \( S = \text{set of natural numbers } \mathbb{N}; \)
  \( A = \{(x, y): x, y \in \mathbb{N} \text{ and } x \leq y\} \)
  Reflexive \( \checkmark; \) Symmetric \( \times; \)
  Antisymmetric \( \checkmark; \) Transitive \( \checkmark \)

• \( S = \{1, 2, 3\}; A = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\} \)
  Reflexive, Symmetric, and Transitive

• \( S = \{x | \text{x is a student in CS230}\}; \)
  \( A = \{(x, y): x, y \in S \text{ and } x \text{ sits in the same row as } y\} \)
  Reflexive, Symmetric, and Transitive
**N-th Power of R**

- Definition: Let \( R \) be any binary relation on \( S \), the \( n \)-th power of \( R \) denoted \( R^n \) is defined as:
  
  \[
  \begin{align*}
  & (a) \ R^0 = \text{Id}, \ (b) \ R^{n+1} = R \circ R^n
  \end{align*}
  \]

- Definition: \( R^+ = R^1 \cup R^2 \cup \ldots \cup R^\infty \)

- Definition: \( R^* = R^0 \cup R^+ \)

- Example: Let \( A = \{a, b, c, d\} \) and \( R = \{(a, b), (b, a), (b, c), (c, d)\} \)

\[
\begin{align*}
  R^0 &= \{(a,a), (b,b), (c,c), (d,d)\} = \text{Id} \\
  R^1 &= R \circ R^0 = R = \{(a,b), (b,a), (b,c), (c,d)\} \\
  R^2 &= R \circ R^1 = \{(a,a), (a,c), (b,b), (b,d)\} \\
  R^3 &= R \circ R^2 = \{(a,b), (a,d), (b,a), (b,c)\} \\
  R^4 &= R \circ R^3 = \{(a,a), (a,c), (b,b), (b,d)\} = R^3 \ (\text{repeat...}) \\
  R^+ &= R^1 \cup R^2 \cup R^3 = \{(a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (b,d), (c,d)\} \\
  R^* &= R^+ \cup R^0 \ . \ i.e. \ also \ include \ (c,c) \ and \ (d,d) \ into \ the \ set
\end{align*}
\]

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**Operation: Closures of Relations**

- Definition: A binary relation \( R^\alpha \) on a set \( S \) is the closure of a relation \( R \) on \( S \) with respect to property \( P \) if

\[
\begin{align*}
  & (a) \ R^\alpha \text{ has property } P, \\
  & (b) \ R \subseteq R^\alpha \text{ and} \\
  & (c) \ R^\alpha \text{ is a smallest relation on } S \text{ that includes } R \\
  \text{and has property } P.
\end{align*}
\]

\[
R^\alpha \leftarrow \text{Closure} (S, R, P)
\]
• Definition: Let \( R \) be any binary relation on \( S \), then

- \( R \cup \text{Id} \) is called **reflexive closure** of \( R \)
- \( R \cup R^{-1} \) is called **symmetric closure** of \( R \)
- \( R^+ \) is called **transitive closure** of \( R \)
- \( R^* \) is called **reflexive & transitive closure** of \( R \)

\[
\begin{align*}
\text{Reflexive closure of } R &= \text{Closure}(S, R, \text{reflexive}) \\
\text{Symmetric closure of } R &= \text{Closure}(S, R, \text{symmetric}) \\
\text{Transitive closure of } R &= \text{Closure}(S, R, \text{transitive}) \\
\text{Reflexive & Transitive closure of } R &= \text{Closure}(S, R, \text{reflexive & transitive})
\end{align*}
\]

Example: Let \( S = \{1,2,3\} \) and \( R = \{(1,1),(1,2),(1,3),(3,1),(2,3)\} \)

**Reflexive closure of \( R \)**
\( R \cup \text{Id} = \{(1,1),(1,2),(1,3),(3,1),(2,3),(2,2),(3,3)\} \)

**Symmetric closure of \( R \)**
\( R \cup R^{-1} = \{(1,1),(1,2),(1,3),(3,1),(2,3),(2,1),(3,2)\} \)

**Transitive closure of \( R \)**
\( R^2 = R \circ R = \{(1,1),(1,2),(1,3),(3,1),(3,2),(3,3),(2,1)\} \)
\( R^3 = R \circ R^2 = \{(1,1),(1,2),(1,3),(3,1),(3,2),(3,3),(2,1),(2,2),(2,3)\} \)

Note: already include all members of \( S \times S \)
\( R^1 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\} = S \times S \)

What is Reflexive & Transitive closure of \( R \)?
• **Definitions:** A binary relation on a set $S$ that is reflexive, symmetric, and transitive is called an **Equivalence Relation** on $S$.

• Examples of Equivalence Relation:

  “Has the same birthday as” on the set of all people

  Logical equivalence of logical sentences

• Example: Let $X = \{1, 2, 3\}; \ R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

• Example: Let $X = \{\text{x| x is a student in CS230}\}; \ R = \{(x, y): x \text{ and } y \text{ sits in the same row and } x, y \in X\}$
  Note: look at each row independently

• Example: On $\mathbb{R}$ (real numbers), define $x \sim y$ if $|x - y| < 0.01$. Then, $\sim$ is reflexive and symmetric, but not transitive
  $\forall x, y \in \mathbb{R}, \ |x - y| = |y - x|$$
  \forall x \in \mathbb{R}, \ |x - 0| < 0.01$
  It is clear that $\sim$ is symmetric
  $\sim$ is reflexive (easy to show all possible cases)
  Let $x = 0.0, y = 0.0075, z = 0.015, x \sim y$ and $y \sim z$, but not $x \sim z$.
Equivalence Class

- Definition: Let $R$ be an Equivalence Relation on a set $A$. The **Equivalence Class** of $A$, denoted $[x]$, is defined by the set of all elements that occur with an element $x \in A$. Thus, 

$$[x] = \{ y \mid y \in A \land [(x,y) \in R \lor (y,x) \in R] \}$$

- Set of all elements in $A$ which appear together with $x$ in pairs of the given Equivalent Relation $R$

$$X = \{ \text{all laptops of ours} \}$$

$$[\text{kg|k}] = \{ \text{all black laptops} \} \subseteq X$$

$$[\text{ks}] = \{ \text{all grey laptops} \}$$

- Examples

If $X$ is the set of all laptops, and $\sim$ is the Equivalence Relation "has the same color as", then one particular Equivalence Class consists of all black laptops.

Example: Let us assume that Eve and Ken live in SJ, Steph and Max live in SF, and Jen lives in SC.

Let $R$ be the equivalence relation $\{(a, b) \mid a$ and $b$ live in the same city$\}$ on the set $P = \{Eve, Ken, Steph, Max, Jen\}$.

Then $R = \{(Eve, Eve), (Eve, Ken), (Ken, Eve), (Ken, Ken), (Steph, Steph), (Steph, Max), (Max, Steph), (Max, Max), (Jen, Jen)\}$.

Then the Equivalence Classes of $R$ are:

$[Eve] = [Ken] = \{Eve, Ken\}$, $\text{in SJ}$

$[Max] = [Steph] = \{Steph, Max\}$, $\text{in SF}$

$[Jen] = \{Jen\}$, $\sim \text{in SC}$
Another example: Let $R$ be the relation: a pair of integers whose difference is a multiple of 3;
\[ R \subseteq \mathbb{Z}^2: (a, b) | a, b \in \mathbb{Z} \text{ and } a \equiv b \pmod{3} \] where “$a \equiv b \pmod{3}$” means $(a-b) = 3 \cdot m$ for some integer $m$

Note: This is called “$a$ is congruent to $b$ modulo 3”.

Example members:
\[(0,0),(0,3),(3,-9),(3,3),(1,4),(-2,7),(4,1),(2,5),(-4,8),\ldots\]

Is $R$ an equivalence relation?
Yes, $R$ is reflexive, symmetric, and transitive.

What are the set of all equivalence classes of $R$?
\[
\{\ldots, -6, -3, 0, 3, 6, \ldots\}, \quad \{\ldots, -5, -2, 1, 4, 7, \ldots\}, \quad \{\ldots, -4, -1, 2, 5, 8, \ldots\}\]

Definition: A **Partition** of a set $S$ is a collection of nonempty disjoint subsets of $S$ whose union equals $S$.

Examples: Let $S$ be the set \{u, m, b, r, o, c, k, s\}. Are the following collections of sets partition of $S$?

\[
\begin{align*}
\{\{m, o, c, k, r\}, \{r, u, b, s\}\} & \quad \text{no} \\
\{\{m, o, c, k\}, \{r, u, b, s\}\} & \quad \text{yes} \\
\{\{c, o, m, b\}, \{u, s\}, \{r\}\} & \quad \text{no} \\
\{\{b, r, o, c, k\}, \{m, u, s, t\}\} & \quad \text{no} \\
\{\{u, m, b, r, o, c, k, s\}\} & \quad \text{yes} \\
\{\{b, o, k\}, \{r, u, m\}, \{c, s\}\} & \quad \text{yes} \\
\{\{u, m, b\}, \{r, o, c, k, s\}, \emptyset\} & \quad \text{no} \\
\{\{u\}, \{m\}, \{b\}, \{r\}, \{o\}, \{c\}, \{k\}, \{s\}\} & \quad \text{yes}
\end{align*}
\]
Theorem: Let \( R \) be an Equivalence Relation on a set \( S \). Then the set of all distinct Equivalence Classes of \( R \) form a Partition of \( S \). Conversely, given a Partition \( \{A_i | i \in I\} \) of the set \( S \), there is an Equivalence Relation \( R \) that has the sets \( A_i, i \in I \), as its Equivalence Classes.

Example: Given partition \( A_1 = \{1, 2, 3\} \) and \( A_2 = \{4, 5\} \).

The corresponding \( R \) contains \((a, b) \) iff \( a \) and \( b \) in the same set of partition.

\[ R = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\} \]

Check for reflexive, symmetric and transitive!!

Definition: A binary relation on a set \( S \) that is reflexive, antisymmetric, and transitive is called a partial ordering on \( S \).

Examples:
1. \( \{(x,y) : \text{where } x \leq y, x, y \in \mathbb{N}\} \)
2. \( \{(x,y) : \text{where } x \subseteq y, x, y \in P \text{ and } P = \text{power set of } \{1,2,3\}\} \)
   \[ P = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} \]
   Some members: \( (\{1\}, \{1\}),(\{1\},\{1,2\}),(\{2,3\}, \{1,2,3\}) \text{ etc} \)
3. The vertex set of a directed acyclic graph ordered by reachability
Partial Orderings and Equivalence Relations

<table>
<thead>
<tr>
<th>Type of Binary Relation</th>
<th>Reflexive</th>
<th>Symmetric</th>
<th>Anti-Symmetric</th>
<th>Transitive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partial Ordering</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Equivalence Relation</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>

**Definition: Relations on Multiple Sets**

Given two sets $S$ and $T$, a binary relation from $S$ to $T$ is a subset of $S \times T$. This means it is possible to have pairs whose left and right members are of different type (yellow page?).

Given $n$ sets $S_1, S_2, \ldots, S_n$, $n > 2$, an $n$-ary relation on $S_1 \times S_2 \times \ldots \times S_n$ is a subset of $S_1 \times S_2 \times \ldots \times S_n$. 

\[
\left\{ (s_1, s_2, \ldots, s_n) \mid s_i \in S_i \right\}
\]
Example: Let \( R = \{ (a, b, c) \mid a = 2b \land b = 2c \text{ with } a, b, c \in \mathbb{N} \} \)

What is the degree of \( R \)?

The degree of \( R \) is 3, because its elements are triples.

Is \((2, 4, 8)\) in \( R \)? No.

Is \((4, 2, 1)\) in \( R \)? Yes.

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**Intro. to Relational Databases**

Let us take a look at a type of database representation that is based on relations, namely the **relational data model**.

A database consists of n-tuples called **records**, which are made up of **fields**. These fields are the entries of the n-tuples.

The relational data model represents a database as an \( n \)-ary relation, that is, a set of records.
Example: Consider a database of students, whose records are represented as 4-tuples with the fields Student Name, ID Number, Major, and GPA:

\[ R = \{ \text{(Ackermann, 231455, CS, 3.88),} \]
\[ \text{(Adams, 888323, Physics, 3.45),} \]
\[ \text{(Chou, 102147, CS, 3.79),} \]
\[ \text{(Goodfriend, 453876, Math, 3.45),} \]
\[ \text{(Rao, 678543, Math, 3.90),} \]
\[ \text{(Stevens, 786576, Psych, 2.99)} \} \]

Relations that represent databases are also called tables, since they are often displayed as tables.

We can apply a variety of operations on n-ary relations to form new relations.

Definition: The projection \( P_{i_1, i_2, ..., i_m} \) maps the n-tuple \((a_1, a_2, ..., a_n)\) to the m-tuple \((a_{i_1}, a_{i_2}, ..., a_{i_m})\), where \( m \leq n \).

i.e. it is used to select some fields from a record.

Example: What is the result when we apply the projection \( P_{2,4} \) to the student record \((\text{Stevens, 786576, Psych, 2.99})\) ?
Solution: It is the pair \((786576, 2.99)\).

In some cases, applying a projection to an entire table may not only result in fewer columns, but also in fewer rows. (select fields from entire table) Why is that?

Some records may only have differed in those fields that were deleted, so they become identical, and there is no need to list identical records more than once.
We can use the **join** operation to **combine two tables** into one if they share some identical fields.

**Definition:** Let \( R \) be a relation of degree \( m \) and \( S \) a relation of degree \( n \). The **join** \( J_p(R, S) \), where \( p \leq m \) and \( p \leq n \), is a relation of degree \( m + n - p \) that consists of all \((m + n - p)\)-tuples \((a_1, a_2, ..., a_{m-p}, c_1, c_2, ..., c_p, b_1, b_2, ..., b_{n-p})\), where the \( m \)-tuple \((a_1, a_2, ..., a_{m-p}, c_1, c_2, ..., c_p)\) belongs to \( R \) and the \( n \)-tuple \((c_1, c_2, ..., c_p, b_1, b_2, ..., b_{n-p})\) belongs to \( S \).

In other words, to generate \( J_p(R, S) \), we have to find all the elements in \( R \) whose \( p \) last components match the \( p \) first components of an element in \( S \). The new relation contains exactly these matches, which are combined to tuples that contain each matching field only once. (See algorithm in textbook)

**Example:** What is \( J_1(Y, R) \), where \( Y \) contains the fields **Year of Birth** and **Student Name**, \( Y = \{(1978, Ackermann), (1972, Adams), (1917, Chou), (1984, Goodfriend), (1982, Rao), (1970, Stevens)\}, \) and \( R \) contains the student records as defined before? \( R = \{(Ackermann, 231455, CS, 3.88), (Adams, 888323, Physics, 3.45), (Chou, 102147, CS, 3.79), (Goodfriend, 453876, Math, 3.45), (Rao, 678543, Math, 3.90), (Stevens, 786576, Psych, 2.99)\} \)


Since \( Y \) has two fields and \( R \) has four, the relation \( J_1(Y, R) \) has \( 2 + 4 - 1 = 5 \) fields.