Overview

- HW#3 Due Now & Pickup HW4 handout!
- Midterm Test #1 (Set, Proof, Logic) is in one week!
- Last Lecture
  - Binary Relations: Set of pairs, A subset of SxS
  - Operations: Inverse, Compositions
  - Types: Identity, Reflexive, Irreflexive, Symmetric, Antisymmetric,
- This Lecture
  - Transitive
  - N\textsuperscript{th} power of R
  - Closure
  - Equivalence Relations, Equivalence Class
  - Partition
  - Partial Ordering
  - Relations with multiple sets
  - Relational Database

N-th Power of R

- Definition: Let R be any binary relation on S, the n\textsuperscript{th} power of R denoted \( R^n \) is defined as:
  
  \begin{align*}
  (a) \, R^0 &= \text{Id}, \quad (b) \, R^{n+1} &= R \circ R^n
  \end{align*}

- Definition : \( R^+ = R^1 \cup R^2 \cup \ldots \cup R^n \)
- Definition : \( R^* = R^0 \cup R^+ = \text{Id} \cup R^+ \)
- Example: Let \( A = \{a,b,c,d\} \) and \( R = \{(a,b), (b,a), (b,c), (c,d)\} \)

  \begin{align*}
  R^0 &= \{(a,a), (b,b), (c,c), (d,d)\} = \text{Id} \\
  R^1 &= R \circ R^0 = R = \{(a,b), (b,a), (b,c), (c,d)\} \\
  R^2 &= R \circ R^1 = \{(a,a), (a,c), (b,b), (b,d)\} \\
  R^3 &= R \circ R^2 = \{(a,b), (a,d), (b,a), (b,c)\} \\
  R^4 &= R \circ R^3 = \{(a,a), (a,c), (b,b), (b,d)\} = R^2 \text{ (repeat...)} \\
  R^+ &= R^1 \cup R^2 \cup R^3 \\
  &= \{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,d)\} \\
  R^* &= R^+ \cup R^0 \text{, i.e. also include (c,c) and (d,d) into the set}
  \end{align*}
Operation: Closures of Relations

- Definition: A binary relation $R^\alpha$ on a set $S$ is the closure of a relation $R$ on $S$ with respect to property $P$ if
  
  (a) $R^\alpha$ has property $P$,
  (b) $R \subseteq R^\alpha$ and
  (c) $R^\alpha$ is a smallest relation on $S$ that includes $R$ and has property $P$.

\[ R^\alpha \leftarrow \text{Closure}(S, R, P) \]

- Definition: Let $R$ be any binary relation on $S$, then

  $R \cup \text{Id}$ is called reflexive closure of $R$
  $R \cup R^{-1}$ is called symmetric closure of $R$
  $R^+$ is called transitive closure of $R$
  $R^*$ is called reflexive & transitive closure of $R$

\[
R \cup \text{Id} = \text{Closure}(S, R, \text{reflexive}) \\
R \cup R^{-1} = \text{Closure}(S, R, \text{symmetric}) \\
R^+ = \text{Closure}(S, R, \text{transitive}) \\
R^* = \text{Closure}(S, R, \text{reflexive & transitive})
\]
Example: Let \( S = \{1,2,3\} \) and \( R = \{(1,1),(1,2),(1,3),(3,1),(2,3)\} \)

Reflexive closure of \( R \)
\[
\mathcal{R}^+ = \mathcal{R} \cup \mathcal{R}^2
\]
\[
\{(1,1),(1,2),(1,3),(3,1),(2,3),(2,2),(3,3)\}
\]

Symmetric closure of \( R \)
\[
\mathcal{R} = \mathcal{R} \cup \mathcal{R}^T
\]
\[
\{(1,1),(1,2),(1,3),(3,1),(2,3),(2,1),(3,2)\}
\]

Transitive closure of \( R \)
\[
R^2 = R \circ R^1 = \{(1,1),(1,2),(1,3),(3,1),(3,2),(3,3),(2,1)\}
\]
\[
R^3 = R \circ R^2 = \{(1,1),(1,2),(1,3),(3,1),(3,2),(3,3),(2,1),(2,2),(2,3)\}
\]
Note: already include all members of \( S \times S \)
\[
R^* = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}
\]

What is Reflexive & Transitive closure of \( R \)?
\[
\mathcal{R}^+ = \bigcup_{n=0}^{\infty} R^n \]

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**Definitions**: A binary relation on a set \( S \) that is **reflexive**, **symmetric**, and **transitive** is called an **Equivalence Relation** on \( S \).

**Examples of Equivalence Relation**:

“Has the same birthday as” on the set of all people

Logical equivalence of logical sentences
• Example: Let $X = \{1,2,3\}; \ R = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

• Example: Let $\ X= \{x|x \text{ is a student in CS230}\};$
  $\ R = \{(x,y) : x, y \text{ sits in the same row and } x,y \in X\}$
  note: look at each row independently

• Example: On $\mathbb{R} (\text{real numbers}),$ define $x \sim y \text{ if } |x - y| < 0.01.$
  Then, $\sim$ is reflexive and symmetric, but not transitive
  It is clear that $\sim$ is symmetric
  $\sim$ is reflexive (easy to show all possible cases)
  Let $x=0.0,$ $y=0.0075,$ $z=0.015,$ $x \sim y$ and $y \sim z,$ but not $x \sim z$

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**Equivalence Class**

• Definition: Let $R$ be an Equivalence Relation on a set $A.$
  **Equivalence Class** of $A$, denoted $[x],$ is defined by the set
  of all elements that occur with an element $x \in A.$ Thus,
  $[x] = \{y | y \in A \land [(x,y) \in R \lor (y,x) \in R]\}$

• **Set of all elements in $A$ which appear together with $x$ in
  pairs of the given Equivalent Relation $R$**

• **Examples**

  If $X$ is the set of all bikes, and $\sim$ is the Equivalence
  Relation "has the same color as", then one particular
  Equivalence Class consists of all white bikes.
• Example: Let us assume that Suzanne and George live in SJ, Stephanie and Max live in SF, and Jennifer lives in SC. Let R be the equivalence relation \{(a, b) \mid a \text{ and } b \text{ live in the same city}\} on the set \(P = \{\text{Suzanne, George, Stephanie, Max, Jennifer}\}\).

Then \(R = \{(\text{Suzanne, Suzanne}), (\text{Suzanne, George}), (\text{George, Suzanne}), (\text{George, George}), (\text{Stephanie, Stephanie}), (\text{Stephanie, Max}), (\text{Max, Stephanie}), (\text{Max, Max}), (\text{Jennifer, Jennifer})\}\).

Then the Equivalence Classes of \(R\) are:
- \([\text{Suzanne}] = [\text{George}] = \{\text{Suzanne, George}\}\),
- \([\text{Max}] = [\text{Stephanie}] = \{\text{Stephanie, Max}\}\),
- \([\text{Jennifer}] = \{\text{Jennifer}\}\).

• Another example: Let \(R\) be the relation a pair of integers whose difference is a multiple of 3; \(\{(a, b) \mid a, b \in \mathbb{Z} \text{ and } a \equiv b \pmod{3}\}\) where “\(a \equiv b \pmod{3}\)” means \((a-b) = 3*m\) for some integer \(m\).

Note: This is called “\(a\) is congruent to \(b\) modulo 3”.

Example members:
- \((0,0),(0,3),(3,-9),(3,3),(1,4),(-2,7),(4,1),(2,5),(-4,8),...\)

Is \(R\) an equivalence relation?

Yes, \(R\) is reflexive, symmetric, and transitive.

What are the set of all equivalence classes of \(R\)?

\[
\{\ldots, -6, -3, 0, 3, 6, \ldots\},
\{\ldots, -5, -2, 1, 4, 7, \ldots\},
\{\ldots, -4, -1, 2, 5, 8, \ldots\}\]
Definition: A **Partition** of a set S is a collection of nonempty disjoint subsets of S whose union equals S.

Examples: Let S be the set \{u, m, b, r, o, c, k, s\}. Do the following collections of sets partition S?

- \{\{m, o, c, k\}, \{r, u, b, s\}\} yes
- \{\{c, o, m, b\}, \{u, s\}, \{r\}\} no ‘k’ is missing
- \{\{b, r, o, c, k\}, \{m, u, s, t\}\} no ‘t’ is not a member of S
- \{\{u, m, b, r, o, c, k, s\}\} yes
- \{\{b, o, k\}, \{r, u, m\}, \{c, s\}\} yes
- \{\{u, m, b\}, \{r, o, c, k, s\}, \emptyset\} no empty set is not allowed
- \{\{u, m, b, r, o, c, k, s\}\} yes

**Theorem:** Let R be an Equivalence Relation on a set S. Then the set of all distinct Equivalence Classes of R form a **Partition** of S. Conversely, given a Partition \{A_i | i \in I\} of the set S, there is an Equivalence Relation R that has the sets A_i, i \in I, as its Equivalence Classes.

Example: Given partition A1=\{1,2,3\} and A2=\{4,5\}.

**The corresponding R contains (a, b) iff a and b in the same set of partition.**

i.e. (1,1),(1,2),(1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3) // for A1  
(4,4),(4,5),(5,4),(5,5) // for A2 belong to R

R = \times \cup \forall

Check for reflexive, symmetric and transitive!!
Definition: A binary relation on a set $S$ that is reflexive, antisymmetric, and transitive is called a **partial ordering** on $S$.

Examples:
1. $\{(x,y) : \text{where } x \leq y, x, y \in \mathbb{N}\}$
2. $\{(x,y) : \text{where } x \subseteq y, x, y \in P \text{ and } P = \text{power set of } \{1,2,3\}\}$
   
   $$P=\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

   Some members: $(\{1\}, \{1\})$, $(\{1\}, \{1,2\})$, $(\{2,3\}, \{1,2,3\})$ etc
3. The vertex set of a directed acyclic graph ordered by reachability

**Partial Orderings and Equivalence Relations**

<table>
<thead>
<tr>
<th>Type of Binary Relation</th>
<th>Reflexive</th>
<th>Symmetric</th>
<th>Anti-Symmetric</th>
<th>Transitive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partial Ordering</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Equivalence Relation</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>
Definition: Relations on Multiple Sets

Given two sets $S$ and $T$, a **binary relation from $S$ to $T$** is a subset of $S \times T$. *(making it possible to have pairs whose left and right members are of different type → yellow page?)*

Given $n$ sets $S_1, S_2, \ldots S_n$, $n > 2$, an **$n$-ary relation on** $S_1 \times S_2 \times \ldots \times S_n$ is a subset of $S_1 \times S_2 \times \ldots \times S_n$.

Example: Let $R = \{(a, b, c) \mid a = 2b \wedge b = 2c \text{ with } a, b, c \in \mathbb{N}\}$

What is the **degree** of $R$?

The degree of $R$ is 3, because its elements are triples.

Is $(2, 4, 8)$ in $R$? No.

Is $(4, 2, 1)$ in $R$? Yes.
Intro. to Relational Databases

Let us take a look at a type of database representation that is based on relations, namely the relational data model.

A database consists of n-tuples called records, which are made up of fields. These fields are the entries of the n-tuples.

The relational data model represents a database as an n-ary relation, that is, a set of records.

Example: Consider a database of students, whose records are represented as 4-tuples with the fields Student Name, ID Number, Major, and GPA:

\[
R = \{ \text{(Ackermann, 231455, CS, 3.88),} \\
\text{(Adams, 888323, Physics, 3.45),} \\
\text{(Chou, 102147, CS, 3.79),} \\
\text{(Goodfriend, 453876, Math, 3.45),} \\
\text{(Rao, 678543, Math, 3.90),} \\
\text{(Stevens, 786576, Psych, 2.99)} \}
\]

Relations that represent databases are also called tables, since they are often displayed as tables.

We can apply a variety of operations on n-ary relations to form new relations.
**Definition:** The projection $P_{i_1, i_2, \ldots, i_m}$ maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$, where $m \leq n$.

i.e. it is used to select some fields from a record.

Example: What is the result when we apply the projection $P_{2,4}$ to the student record (Stevens, 786576, Psych, 2.99)?

Solution: It is the pair (786576, 2.99).

In some cases, applying a projection to an entire table may not only result in fewer columns, but also in fewer rows.

(Select fields from entire table) Why is that?

Some records may only have differed in those fields that were deleted, so they become identical, and there is no need to list identical records more than once.

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We can use the join operation to combine two tables into one if they share some identical fields.

**Definition:** Let $R$ be a relation of degree $m$ and $S$ a relation of degree $n$. The join $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to $R$ and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to $S$.

In other words, to generate $J_p(R, S)$, we have to find all the elements in $R$ whose $p$ last components match the $p$ first components of an element in $S$. The new relation contains exactly these matches, which are combined to tuples that contain each matching field only once.

(See algorithm in textbook)
Example: What is $J_1(Y, R)$, where $Y$ contains the fields **Year of Birth** and **Student Name**, 

$Y = \{(1978, \text{Ackermann}), (1972, \text{Adams}), (1917, \text{Chou}), (1984, \text{Goodfriend}), (1982, \text{Rao}), (1970, \text{Stevens})\}$,

and $R$ contains the student records as defined before?

Solution: The resulting relation is:

$$R = \{(\text{Ackermann}, 231455, \text{CS}, 3.88),$$
$$\quad (\text{Adams}, 888323, \text{Physics}, 3.45),$$
$$\quad (\text{Chou}, 102147, \text{CS}, 3.79),$$
$$\quad (\text{Goodfriend}, 453876, \text{Math}, 3.45),$$
$$\quad (\text{Rao}, 678543, \text{Math}, 3.90),$$
$$\quad (\text{Stevens}, 786576, \text{Psych}, 2.99)\}$$

Since $Y$ has two fields and $R$ has four, the relation $J_1(Y, R)$ has $2 + 4 - 1 = 5$ fields.