Overview

• HW3 Due & HW4 assignment will be online!
• HW4 is due in 2 weeks on 10/13 Tuesday
• Midterm #1 (Set, Proof, Logic) is in one week!
• Study for Midterm #1:
  – To guide your study for MT1, I will give a review lecture this Thursday, where I go through the answers for HW1, 2 and 3
• Last lecture: Completed Formal Logics
  – Predicate Logic
  – Universal and Existential Quantifiers
  – Predicate to English Translation
  – English to Predicate Translation
  – Equivalence Laws with Quantifiers
  – Inference Rules with Quantifiers
  – Deductive Proofs with Predicate

• This lecture: Relations
  – Binary Relation

Covers materials from the first to the last lectures:
1) Video always on
2) Do not leave

3. Relation

CSC230
A relation is used to describe certain properties of various things.

Many applications:
- Business and its phone number
- Employee and salary
- Relation between data tables
- $x$ and function $f(x)$
- Graphs

**Binary Relations: Pairs**

**Definition:** Given a set $S$, a binary relation on $S$ is a subset of $S \times S$. $S \times S$ is the Cartesian product of a set $S$ with itself, i.e., the set of all ordered pairs of elements of $S$.

$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$

$\emptyset \subseteq A \times A$

Example: Let $S = \{1, 2, 4\}$.

$S \times S = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (4, 1), (4, 2), (4, 4)\}$

$R = \{(x, y) \mid x, y \in S \text{ and } x = y/2\} = \{(1, 2), (2, 4)\} \subseteq S \times S$
Example: Let \( S = \{1, 2, 3\} \).
\[
S \times S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}
\]

**Identity relation (Equality relation)**

Example: \( R = \{(x, y) \mid x, y \in S \text{ and } x = y\} = \{(1, 1), (2, 2), (3, 3)\} \)

Notation: use \( \text{Id} \) to denote identity relation

Q: \( R = \{(1, 1), (2, 2)\} \) ? \( S = \{1, 2, 3\} \)?

**Empty/Void/Trivial relation**

\( \forall \text{ No Pair} \quad \emptyset \subseteq S \times S \)

Example: \( R = \emptyset \)

**Universal relation**

\( \forall \text{ All Pair} \quad S \times S \subseteq S \times S \)

Example: \( R = \{(x, y) \mid x, y \in S\} = S \times S \)

Also, “less than”, “greater than”, “less than and equal to”, etc

\[ |S \times S| = |S| \times |S| = 4 \times 4 = 16 \]

- **Example:** Let \( S = \{1, 2, 3, 4\} \).
  
  \[
  R = \{(a, b) \mid a, b \in S \text{ and } a < b\} \\
  \]
  
  \[
  \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}
  \]

Q: replacing \( a < b \) by \( a = b \)?
  
  \( a \leq b \)?
Question: How many different relations can we define on a set $A$ with $n$ elements?

A relation on a set $A$ is a subset of $A \times A$.

How many elements are in $A \times A$? 

There are $n^2$ elements in $A \times A$, so how many subsets (= relations on $A$) does $A \times A$ have? (What is the cardinality of the power set of $A \times A$?)

The number of all possible subsets that we can form out of a set with $k$ elements is $2^k$. Therefore, $2^{n^2}$ subsets can be formed out of $A \times A$.

Answer: We can define $2^{n^2}$ different relations on $A$. 

Operations for binary relations:

- **Inverse (unary op):** Let $R$ be a binary relation. The inverse of $R$, denoted $R^{-1}$, is $\{(y,x) \mid (x,y) \in R\}$

  Example: $R = \{(1,0),(2,3),(1,3),(2,1),(1,1)\}$

  $R^{-1} = \{(0,1),(3,2),(3,1),(1,2),(1,1)\}$

  Note: $(R^{-1})^{-1} = R$; $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$; if $S \subseteq R$ then $S^{-1} \subseteq R^{-1}$

- **Composition (binary op):** Let $R$ and $S$ be binary relations on the same set $X$. The composition of $R$ and $S$, denoted $R \circ S$, is:

  $R \circ S = \{(x,y) \mid (x,z) \in S \text{ and } (z,y) \in R\}$ (note the order!)

  Example: Let $R = \{(1, 2), (3, 5), (3, 7), (4, 6), (6, 8), (7, 10)\}$ and $S = \{(2, 4), (3, 6), (5, 7), (7, 9), (4, 9)\}$

  $R \circ S = \{(2, 6), (3, 8), (5, 10)\}$
**Properties (Types) of Relations:**

Definition: A relation $R$ on a set $A$ is called **reflexive** if $(a, a) \in R$ for every element $a \in A$, i.e. $R$ is reflexive iff $Id \subseteq R$.

Examples of reflexive relations include:
- "is equal to"
- "is a subset of"
- "is greater/less than or equal to"

Example: Are the following relations on \{1, 2, 3, 4\} reflexive?

- $R_1 = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$
- $R_2 = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$
- $R_3 = \{(1, 1), (2, 2), (3, 3)\}$

Definition: A relation $R$ on a set $A$ is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.

i.e. $R$ is irreflexive iff $R \cap Id = \emptyset$.

Examples of irreflexive relations include:
- "is not equal to"
- "is a proper subset of"
- "is greater than"
Definitions:
• A relation $R$ on a set $A$ is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$, i.e. $R$ is symmetric iff $R = R^{-1}$.

• A relation $R$ on a set $A$ is called **antisymmetric** if $(a, b) \in R$ implies $(b, a) \not\in R$ for all $a \neq b, a, b \in A$, i.e. $R$ is antisymmetric iff $R \cap R^{-1} \subseteq \text{Id}$.

• Examples of symmetric relation:
  – "is married to" $x \text{ is married to } y \iff y \text{ is married to } x$
  – "is equal to" $x = y \iff y = x$
  – “$x$ is odd and $y$ is odd too“
  – The identity relation

• Examples of antisymmetric relation
  – The identity relation
  – $x \leq y \iff \neg x < y$
  – $x$ is subset of $y \iff \neg x \subset y$
  – "$x$ is even, $y$ is odd"

Example: Are the following relations on $\{1, 2, 3, 4\}$ reflexive or irreflexive or symmetric or antisymmetric?

- $R1 = \{(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)\}$ symmetric
- $R2 = \{(1, 1)\}$ sym. and antisym
- $R3 = \{(1, 3), (3, 2), (2, 1)\}$ antisym and irreflexive
- $R4 = \{(4, 4), (3, 3), (1, 4)\}$ antisym.
- $R5 = \text{Id}$ reflexive, sym, antisym
• Definition: A relation $R$ on a set $A$ is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

i.e. $R$ is transitive iff $R \circ R \subseteq R$.

• Examples of transitive relations:
  - "is a subset of" $A \subseteq B, B \subseteq C \rightarrow A \subseteq C$
  - "divides" $a \mid b, b \mid c \rightarrow a \mid c$
  - "implies" $a \rightarrow b, b \rightarrow c \rightarrow a \rightarrow c$
  - “path in graph”

Example: Are the following relations on $\{1, 2, 3, 4\}$ transitive?

$R_1 = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\}$

$R_2 = \{(1, 3), (3, 2), (2, 1)\}$

$R_3 = \{(2, 4), (4, 3), (2, 3), (4, 2)\}$

• Example: Test each binary relation on the given set $S$ for reflexivity, irreflexivity, symmetry, antisymmetry, and transitivity. (next page….)
• $S = \text{set of natural numbers } \mathbb{N};$
  
  $A = \{(x, y): \ x, y \in \mathbb{N} \text{ and } x \leq y\}$

  Reflexive $\checkmark$; Symmetric $\times$;

  Antisymmetric $\checkmark$; Transitive $\checkmark$

• $S = \{1, 2, 3\}; \ A = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

  Reflexive, Symmetric, and Transitive

• $S = \{x \mid x \text{ is a student in CS230}\};$
  
  $A = \{(x, y): x, y \in S \text{ and } x \text{ sits in the same row as } y\}$

  Reflexive, Symmetric, and Transitive