Formal Logic 4

• HW#3 due on Oct 1st Tuesday. Work on it!
• Schedule up to the first midterm
  • Last lecture
    – Deductive Proofs
    – Predicate Logic
    – Universal and Existential Quantifiers
    – Predicate to English Translation

  • This lecture
    – English to Predicate Translation
    – Equivalence Laws with Quantifiers
    – Inference Rules with Quantifiers
    – Deductive Proofs with Predicate

Syntax of
• A predicate is a function, in general, has the form

  \[ P(x_1, x_2, ..., x_n) \]

  which maps from \( x_1, x_2, ..., x_n \) to the values true and false

  where \( P \) is the name of the predicate,
  \( x_i \) are variables or parameters
  \( n \) is the degree of the predicate.
• Proposition is simply a statement that is either true or false, has no variables involved.

\[ P(x) : x < 777 \]

• But predicates can take variables, and once we replace the variable by a constant (instantiate), it becomes a proposition.

\[ \text{Predicate Instantiation Proposition} \]

• Introduce two quantifiers:
  
  - Universal quantifier \( \forall \) (for all),
  - Existential quantifier \( \exists \) (exists)

A nonempty set \( U \) which is called universe or domain.

\[ \forall x \ P(x) \]
- is true if \( P(x) \) is true for every \( x \) in \( U \)
- otherwise, false.

\[ \exists x \ P(x) \]
- is true if \( P(x) \) is true for at least one \( x \) in \( U \)
- is false if \( P(x) \) is false for every \( x \) in \( U \)

• \( P(x) \) is a predicate.

• \( \forall x \ P(x) \), \( \exists x \ P(x) \) : either true or false, so they are proposition

\[ \text{Predicates Quantified by Proposition} \]
Example:

\( \text{U = all birds} \)

\( \text{P(x): x is a peacock} \)

\( \text{T(x): x is proud of its tail} \)

\( \text{S(x): x can sing} \)

Some birds that are proud of their tails cannot sing.

\[ \exists x \left[ T(x) \land \neg S(x) \right] \]

Some peacocks cannot sing.

\[ \exists x \left[ P(x) \land \neg S(x) \right] \]

No birds are proud of their tails.

\[ \neg \exists x \left[ T(x) \right] \]

No birds, except peacocks, are proud of their tails.

\[ \neg \exists x \left[ \neg P(x) \land T(x) \right] \equiv \forall x \left[ \neg P(x) \rightarrow \neg T(x) \right] \]

Example:

\( \text{U = the whole world} \)

\( \text{B(x): x is a bee} \)

\( \text{F(x): x is a flower} \)

\( \text{L(x, y): x loves y} \)

All bees love all flowers:

\[ \forall x \forall y \left[ B(x) \rightarrow F(y) \rightarrow L(x, y) \right] \] 

\[ \equiv \forall x \forall y \left[ B(x) \rightarrow F(y) \right] \land \neg L(x, y) \rightarrow Q \rightarrow R \]

Every bee loves only flowers:

\[ \forall x \left[ \forall y \left[ B(x) \rightarrow F(y) \land \neg L(x, y) \right] \right] \] 

\[ \equiv \forall x \left[ B(x) \rightarrow F(x) \right] \land \neg L(x, y) \rightarrow Q \rightarrow R \]
Example: $Q(x)$: true if $x$ is a rational number

\[ \{ x \mid x = \frac{n}{m}, n \in \mathbb{Z}, m \in \mathbb{Z}, m \neq 0 \} \]

\[ \forall x \forall y \left( Q(x) \land Q(y) \land (x < y) \right) \rightarrow \exists u \left( Q(u) \land (x < u) \land (u < y) \right) \]

“There is a rational number in between every pair of distinct rational numbers”

Note: May define a function for $(a < b)$

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**Inference rules for predicate logic**

- All inference rules in propositional logic

- $\forall x \ P(x) \rightarrow P(c)$ (universal instantiation)
  
  For any $c$ in $U$, $c$ may be a variable

- $\exists x \ P(x) \rightarrow P(c)$ (existential instantiation)
  
  $c$ is a member of $U$ such that $P(c) = True$, $c$ is a constant
  
  Note: when $c$ is used in multiple proof steps, need to make sure it is valid (the same $c$ cannot be reused).

- $P(c) \rightarrow \forall x \ P(x)$ (universal generalization)
  
  $P(c)$ is true for arbitrary member $c$ in $U$

- $P(c) \rightarrow \exists x \ P(x)$ (existential generalization)
  
  $c$ is a member of $U$ such that $P(c) = True$

- There are more restrictions in these rules, see notes in http://www.cs.odu.edu/~toida/nerzic/content/logic/pred_logic/inference/infer_intro.html
Example: Every SFSU student is kind. George is a SFSU student. Therefore, George is kind. Is this valid?

\[ \forall x \ (S(x) \rightarrow G(x)) \]

S(x): “x is a SFSU student.”
G(x): “x is kind.”

Want to show:

\[ (\forall x \ (S(x) \rightarrow G(x))) \land S(\text{George}) \rightarrow G(\text{George}) \]

S1: \( \forall x \ (S(x) \rightarrow G(x)) \) Hypothesis
S2: \( S(\text{George}) \) Hypothesis
S3: \( S(\text{George}) \rightarrow G(\text{George}) \) 1, Univ. instantiation
S4: \( G(\text{George}) \) 2 & 3, Modus ponens

Example: Prove deductively

\[ \forall x \ [P(x) \land Q(x)] \rightarrow \forall x \ P(x) \land \forall x \ Q(x) \]

S1. \( \forall x \ [P(x) \land Q(x)] \) Hypothesis
S2. \( P(x) \land Q(x) \) S1, Univ. Inst.
S3. \( P(x) \) S2, simplification
S4. \( Q(x) \) S2, simplification
S5. \( \forall x \ P(x) \) S3, Univ. Gen.
S6. \( \forall x \ Q(x) \) S4, Univ. Gen.
S7. \( \forall x \ P(x) \land \forall x \ Q(x) \) S5, S6, conjunction
Example:

$$(\exists x)P(x) \land (\exists x)Q(x) \rightarrow (\exists x)[P(x) \land Q(x)]$$

a. Find an interpretation to prove this wff is not valid.

Ans: 

$P(x)$: $x$ is even, $Q(x)$: $x$ is odd

b. What's wrong in the following proof sequence?

S1. $(\exists x)P(x)$ Hypothesis
S2. $(\exists x)Q(x)$ Hypothesis
S3. $P(a)$ S1, Exist. Inst
S4. $Q(a)$ S2, Exist. Inst
S5. $P(a) \land Q(a)$ S3, S4 conjunction
S6. $(\exists x)[P(x) \land Q(x)]$ S5, Exist. Gen.