Formal Logic 4

• HW#3 due on upcoming Tuesday. Work on it.
• Schedule up to the first midterm
  • Last lecture
    – Deductive Proofs
    – Predicate Logic
    – Universal and Existential Quantifiers
    – Predicate to English Translation
  • This lecture: Completing Formal Logics & More...
    – English to Predicate Translation
    – Equivalence Laws with Quantifiers
    – Inference Rules with Quantifiers
    – Deductive Proofs with Predicate
    – Q&A + HW1&2 Answers.

Syntax of Predicate Logic:

• Inherits everything from propositional logic
• A *predicate* is a function, in general, has the form

\[ P(x_1, x_2, ..., x_n) \]

which maps from \( x_1, x_2, ..., x_n \) to the truth values true and false.

where

- \( P \) is the name of the predicate,
- \( x_i \) are variables or parameters
- \( n \) is the degree of the predicate
  = \# of variables
• **Proposition** is simply a declarative statement that is either true or false, has no variables involved.

• But **predicates can take variables**, and once we replace the variable by a constant (**instantiate**), it becomes a proposition. 

• Two quantifiers: (what for?)

  **Universal quantifier** \( \forall \) (for all),
  \[ \forall x P(x) \]
  - is true if \( P(x) \) is true for **every** \( x \) in \( U \)
  - otherwise, false.

  \[ \forall x P(x) \iff P(a_1) \land P(a_2) \land ... \land P(a_{n-1}) \land P(a_n) \]

  **Existential quantifier** \( \exists \) (exists),
  \[ \exists x P(x) \]
  - is true if \( P(x) \) is true for **some/at least** one \( x \) in \( U \)
  - is false if \( P(x) \) is false for every \( x \) in \( U \)

  \[ \exists x P(x) \iff P(a_1) \lor P(a_2) \lor ... \lor P(a_{n-1}) \lor P(a_n) \]

• \( P(x) \) is a **predicate**.

• \( \forall x P(x) \land \exists x P(x) \): either true or false, so they are proposition
Example:

\[ \forall x \rightarrow (\neg P(x) \land \neg T(x)) \]

All birds that are not peacocks are not proud of their tails.

\[ \forall x \rightarrow (\neg P(x) \land \neg T(x)) \]

- Some birds that are proud of their tails cannot sing.
  \[ \exists x [ T(x) \land \neg S(x)] \]

- Some peacocks cannot sing.
  \[ \exists x [ P(x) \land \neg S(x) ] \]

- No birds are proud of their tails.
  \[ \neg \exists x [ T(x) ] \]

- No birds, except peacocks, are proud of their tails.
  \[ \neg \exists x [ P(x) \land \neg T(x) ] \]

Contrapositive law

- All birds that are proud of tail are peacocks.
  \[ \forall x \rightarrow (\neg P(x) \land T(x)) \leftrightarrow \neg \exists x [ \neg P(x) \land \neg T(x) ] \]

- All birds that are not peacocks are not proud of tail.
  \[ \forall x \rightarrow (\neg P(x) \land T(x)) \leftrightarrow \neg \exists x [ P(x) \land \neg T(x) ] \]

Example:

\[ \forall x \rightarrow (\neg P(x) \land T(x)) \]

All birds that are proud of tail are peacocks.

\[ \forall x \rightarrow (\neg P(x) \land T(x)) \]

- All bees love all flowers:
  \[ \forall x [B(x) \rightarrow \forall y [F(y) \rightarrow L(x, y) ]] \]

\[ \forall x \rightarrow (\neg P(x) \land T(x)) \]

- Every bee loves only flowers:
  \[ \forall x [B(x) \rightarrow \forall y [L(x, y) \rightarrow F(y)]] \]

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Example: \(Q(x): \text{true if } x \text{ is a rational number } (\in \mathbb{Q})\)

\[\{ x \mid x = \frac{m}{n}, n \in \mathbb{Z}, a \in \mathbb{Z} \wedge n \neq 0 \}\]

“There is a rational number in between every pair of distinct rational numbers”

\[\forall x, y (Q(x) \land Q(y) \land (x < y) \rightarrow \exists u (Q(u) \land (x < u) \land (u < y)))\]

Note: May define a function for \((a < b)\)

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**Inference rules for predicate logic**

- All inference rules in propositional logic

- \(\forall x P(x) \rightarrow P(c)\) (universal instantiation)
  
  For any \(c\) in \(U\), \(c\) may be a variable

- \(\exists x P(x) \rightarrow P(c)\) (existential instantiation)
  
  \(c\) is a member of \(U\) such that \(P(c) = \text{True}\), \(c\) is a constant

  Note: when \(c\) is used in multiple proof steps, need to make sure it is valid
  
  (the same \(c\) cannot be reused)

- \(P(c) \rightarrow \forall x P(x)\) (universal generalization)
  
  \(P(c)\) is true for arbitrary member \(c\) in \(U\)

- \(P(c) \rightarrow \exists x P(x)\) (existential generalization)
  
  \(c\) is a member of \(U\) such that \(P(c) = \text{True}\)

- There are more restrictions in these rules, see notes in
  
  [http://www.cs.odu.edu/~toida/nerzic/content/logic/pred_logic/inference/infer_intro.html](http://www.cs.odu.edu/~toida/nerzic/content/logic/pred_logic/inference/infer_intro.html)
• Example: Every SFSU student is kind. George is a SFSU student. Therefore, George is kind. Is this valid?

\[ S(x): \text{“}x\text{ is a SFSU student.”} \]
\[ G(x): \text{“}x\text{ is kind.”} \]

Want to show :
\[
(\forall x (S(x) \rightarrow G(x))) \land S(\text{George}) \rightarrow G(\text{George})
\]

\[ \begin{align*}
S1: \quad & \forall x (S(x) \rightarrow G(x)) & \text{Hypothesis} \\
S2: \quad & S(\text{George}) & \text{Hypothesis} \\
S3: \quad & S(\text{George}) \rightarrow G(\text{George}) & 1, \text{Univ. instantiation} \\
S4: \quad & G(\text{George}) & 2 \& 3, \text{Modus ponens}
\end{align*} \]

\[ \square \]

Example: Prove deductively
\[
\forall x [P(x) \land Q(x)] \rightarrow \forall x P(x) \land \forall x Q(x)
\]

\[ \begin{align*}
S1. \quad & \forall x [P(x) \land Q(x)] & \text{Hypothesis} \\
S2. \quad & P(x) \land Q(x) & S1, \text{Univ. Inst.} \\
S3. \quad & P(x) & S2, \text{simplification} \\
S4. \quad & Q(x) & S2, \text{simplification} \\
S5. \quad & \forall x P(x) & S3, \text{Univ. Gen.} \\
S6. \quad & \forall x Q(x) & S4, \text{Univ. Gen.} \\
S7. \quad & \forall x P(x) \land \forall x Q(x) & S5, S6, \text{conjunction}
\end{align*} \]

\[ \square \]
Example:

\[(\exists x)P(x) \land (\exists x)Q(x) \rightarrow (\exists x)[ P(x) \land Q(x) ]\]

a. Find an interpretation to prove this wff is not valid.

Ans: \(P(x): x\) is even, \(Q(x): x\) is odd

b. What's wrong in the following proof sequence?

S1. \((\exists x)P(x)\) Hypothesis
S2. \((\exists x)Q(x)\) Hypothesis
S3. \(P(a)\) S1, Exist. Inst
S4. \(Q(a)\) S2, Exist. Inst
S5. \(P(a) \land Q(a)\) S3, S4 conjunction
S6. \((\exists x)[ P(x) \land Q(x) ]\) S5, Exist. Gen.