Formal Logic 3

• HW#3 due in one week. Work on it!
• Midterm #1 will be in 2 weeks on Mar 9 Tuesday
  – Study HW questions/Example problems in Notes/Textbook
  – Try recreate answers for all examples without seeing them!
  – You have to have your video on for the entire duration. You must make sure your video works!!! Consult me asap if you have/foresee problems.
• Q&A and HW1&2 Ans at the end of next lecture!
• Last lecture: Proof using propositional logic
  – Normal forms
  – Deductive Proofs: Show that “Conditions $\rightarrow$ Conclusion” is a tautology by applying a series of Inference Rules such as Modus Ponens

• This lecture: Cont. of Logical Proofs
  – Deductive Proofs Cond.
  – Predicate Logic

Important Inferences rules (more in the textbook)

$P \rightarrow Q$  Modus Ponens  \hspace{1cm} Extract Conclusion from $\rightarrow$
\[\begin{array}{c}
P \\ \hline \\ Q \end{array}\]

$P \land Q$  Conjunction  \hspace{1cm} Create $\land$
\[\begin{array}{c}
P \\ Q \\ \hline \\ P \land Q \end{array}\]

$P \lor Q$  Disjunctive Syllogism  \hspace{1cm} Extract from $\lor$
\[\begin{array}{c}
\neg Q \\ \hline \\ P \end{array}\]

$P \rightarrow Q$  Modus Tollens  \hspace{1cm} Extract Hypothesis from $\rightarrow$
\[\begin{array}{c}
\neg Q \\ \hline \\ \neg P \end{array}\]

$P \lor Q$  Addition  \hspace{1cm} Create $\lor$
\[\begin{array}{c}
P \\ Q \\ \hline \\ P \lor Q \end{array}\]

$P \land Q$  Simplification  \hspace{1cm} Extract from $\land$
\[\begin{array}{c}
P \\ Q \\ \hline \\ P \land Q \end{array}\]
Example: Prove \((P \lor Q \rightarrow R) \land (R \rightarrow S \lor W) \rightarrow (Q \land \lnot S \rightarrow W)\) is a theorem

Note: By using the conditional proof law, this is equivalent to
\[(P \lor Q \rightarrow R) \land (R \rightarrow S \lor W) \land (Q \land \lnot S) \rightarrow W\]

1. \(P \lor Q \rightarrow R\) (hypothesis)
2. \(R \rightarrow S \lor W\) (hypothesis)
3. \(Q \land \lnot S\) (hypothesis)

** Target: Show \(W\) is true: **

4. \(Q\) (3, simplification)
5. \(P \lor Q\) (4, addition)
6. \(R\) (1, 5 mp)
7. \(S \lor W\) (2, 6 mp)
8. \(\lnot S\) (3, simplification)
9. \(W\) (7, 8 disjunctive syllogism) □

Example: Is the following argument valid (use propositional logic)?

Gary is intelligent or a good actor.
If Gary is intelligent, then he can count from 1 to 10.
Gary can only count from 1 to 3.
Therefore, Gary is a good actor.

\(i: \text{"Gary is intelligent."} \)
\(a: \text{"Gary is a good actor."} \)
\(c: \text{"Gary can count from 1 to 10."} \)

\((i \lor a) \land (i \rightarrow c) \land \lnot c \rightarrow a\)

\((i \lor a) \land (i \rightarrow c) \land \lnot c \rightarrow a\)

\(h1 \quad h2 \quad h3 \quad \text{Conc} \)

\(\text{Conclusion with } \rightarrow\)
Step 1: \(-c\)  
Step 2: \(i \rightarrow c\)  
Step 3: \(i \lor a\)  
Step 4: \(-i\)  
Step 5: \(a\)  

Another example:

If you listen to me, you will pass CS 230.  
You passed CS 230.  
Therefore, you have listened to me.

Is this argument valid?

No, the statement \((p \rightarrow q) \land q\) \(\rightarrow\) \(p\) is not a tautology.  
It is false if \(p\) is false and \(q\) is true.

You did not pass CS230 therefore you have not listened to me?

2.1 Introduction to Predicate Logic

What is Predicate?

- Recall a statement “\(x < 777\)” which is not a proposition. Let’s define a function \(P(x)\) indicating “\(x < 777\)”
- \(P\) is called predicate and \(x\) is the variable
- We can instantiate a predicate by giving a constant value to the variable, making a predicate to a proposition
- Truth value depends on value of variable
  - The truth value of \(P(500)\) is “True” (\(x=500\))
  - The truth value of \(P(1000)\) is “False” (\(x=1000\))

- Other Examples:

  \(\text{greater}(x, y)\) = which is true if \(x > y\); false otherwise

  \(\text{prime}(x)\) = which is true if \(x\) is a prime number, false otherwise
**Syntax of Predicate Logic:**

- All in Propositional Logic (Propositions & Connectives) + Predicates
- A **predicate** is a function, in general, has the form

\[ P(x_1, x_2, ..., x_n) \]

which maps from \( x_1, x_2, ..., x_n \) to the truth values **true and false**.

where

- \( P \) is the name of the predicate,
- \( x_i \) are variables or parameters
- \( n \) is the degree of the predicate
  
  \( = \) # of variables

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- **Proposition** is simply a declarative statement that is either true or false, has no variables involved.

- But **predicates can take variables**, and once we replace the variable by a constant (**instantiate**), it becomes a proposition. **Predicate instantiate** \( P(x) \rightarrow \text{Proposition} \)
  
  Statement about collection of things

- Two quantifiers: (what for?)

  - **Universal quantifier** \( \forall \) (for all),
  - **Existential quantifier** \( \exists \) (exists),

A nonempty set \( U \) which is called **universe or domain**.
<table>
<thead>
<tr>
<th>Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\forall x \ P(x))</td>
<td>(For all (x \ P(x)) is true)</td>
</tr>
<tr>
<td>(\forall x \forall y P(x,y))</td>
<td>- is true if (P(x)) is true for every (x) in (U)</td>
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<tr>
<td></td>
<td>- otherwise, false.</td>
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<td></td>
<td>(\forall x P(x) \iff P(a_1) \land P(a_2) \land \ldots \land P(a_{n-1}) \land P(a_n))</td>
</tr>
<tr>
<td>(\exists x \ P(x))</td>
<td>(There exists (x) such that (P(x)) is true)</td>
</tr>
<tr>
<td>(\exists x \exists y P(x,y))</td>
<td>- is true if (P(x)) is true for at least one/some (x) in (U)</td>
</tr>
<tr>
<td></td>
<td>- is false if (P(x)) is false for every (x) in (U)</td>
</tr>
<tr>
<td></td>
<td>(\exists x P(x) \iff P(a_1) \lor P(a_2) \lor \ldots \lor P(a_{n-1}) \lor P(a_n))</td>
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- \(P(x)\) is a predicate.
- \(\forall x \ P(x)\), \(\exists x \ P(x)\) : either true or false, so they are proposition.

**Example:**

Let \(U = \) all people
\(S(x): x\) is a SFSU student.
\(G(x): x\) is a genius.
\(P(x): x\) is a SFSU professor.

(i) What does \(\forall x (S(x) \rightarrow G(x))\) mean?

“For all \(x\) in \(U\), If \(x\) is a SFSU student, then \(x\) is a genius.” or “All SFSU students are geniuses.”

Note: \(\forall x (S(x) \land G(x))\) means “all people are SFSU students and are genius”
Example:

Let $U =$ all people
$S(x): x$ is a SFSU student.
$G(x): x$ is a genius.
$P(x): x$ is a SFSU professor.

(ii) What does $\exists x \ (P(x) \land G(x))$ mean?

“There is an $x$ in $U$ such that $x$ is a SFSU professor and $x$ is a genius.” or
“At least one SFSU professor is a genius.”
“Some SFSU professor is genius.”

Note: $\exists x \ (P(x) \rightarrow G(x))$
means “There is $x$ in $U$, if $x$ is a SFSU professor, then $x$ is genius”, This is true even for all who is not a SFSU professor!

Example:

Let $U =$ all people
$S(x): x$ is a SFSU student.
$G(x): x$ is a genius.
$P(x): x$ is a SFSU professor.

Let $U =$ all SFSU students
the statement “All SFSU students are geniuses” can be written as $\forall x \ G(x)$ instead of $\forall x \ (S(x) \rightarrow G(x))$
Another example:
Let \( U = \) the real numbers (\( \mathbb{R} \)).
What does \( \forall x \exists y (x + y = 320) \) mean?
“For every \( x \) there exists a \( y \) such that \( x + y = 320 \).”
Q: Is it true?
Q: Is it true for \( U = \) the natural numbers (\( \mathbb{N} \))? 
Q: Counter example?

Note: \( \forall x \exists y (x + y = 320) = \forall x (\exists y (x + y = 320)) \)

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Translate English sentences to predicates
Example:
\( U = \) all reports
\( A(x): x \) is authorized
\( T(x): x \) is trustworthy
\( F(x): x \) is false

Some unauthorized reports are false.
\[ \exists x [\neg A(x) \land F(x)] \leftrightarrow \exists x [F(x) \land \neg A(x)] \]

All authorized reports are trustworthy.
\[ \forall x [A(x) \rightarrow T(x)] \leftrightarrow \forall x [\neg T(x) \rightarrow \neg A(x)] \leftrightarrow \forall x [\neg A(x) \lor T(x)] \]

Some false reports are not trustworthy.
\[ \exists x [F(x) \land \neg T(x)] \leftrightarrow \exists x [\neg T(x) \land F(x)] \]
De Morgan’s law for quantifiers (They are interchangeable!):

Not all of... \( \neg (\forall x \ P(x)) \iff \exists x (\neg P(x)) \)
None of... \( \neg (\exists x \ P(x)) \iff \forall x (\neg P(x)) \)

Example: Not all roses are red
\[ \neg \forall x (\text{Rose}(x) \to \text{Red}(x)) \iff \forall x (\neg \text{Rose}(x) \lor \text{Red}(x)) \]
\[ \exists x (\neg \text{Rose}(x) \lor \text{Red}(x)) \]
Some roses are not red

Example: Nobody is perfect
\[ \neg \exists x (\text{Person}(x) \land \text{Perfect}(x)) \iff \forall x (\neg \text{Person}(x) \lor \neg \text{Perfect}(x)) \]
\[ \forall x \neg \text{Person}(x) \lor \forall x \neg \text{Perfect}(x) \]
Everyone is imperfect