Formal Logic 3

- HW#3 due in one week. Work on it!
- Midterm #1 will be in 2 weeks on 10/6
  - Study HW questions/Example problems in Notes/Textbook
  - Try recreate answers for all examples without seeing them!
  - You have to have your video on for the entire duration. You must make sure your video works!!! Consult me asap if you see problems.
- Q/A for Sets/Proof/Logic at the end of next lecture. Bring questions!!!
- Last lecture: Proof using propositional logic
  - Normal forms
  - Deductive Proofs: Show that “Conditions $\rightarrow$ Conclusion” is a tautology by applying a series of Inference Rules such as Modus Ponens

This lecture: Cont. of Logical Proofs
- Deductive Proofs Cond.
- Predicate Logic

Important Inferences rules (more in the textbook)

- $P \rightarrow Q$ Modus Ponens
  - $P$
  - $\rightarrow$
  - $Q$

- $P$ Conjunction
  - $Q$
  - $\wedge$
  - $P \wedge Q$

- $P \lor Q$ Disjunctive Syllogism
  - $\neg Q$
  - $\lor$
  - $P$

- $P \rightarrow Q$ Modus Tollens
  - $\neg Q$
  - $\rightarrow$
  - $\neg P$

- $P$ Addition
  - $Q$
  - $\lor$
  - $P \lor Q$

- $P \land Q$ Simplification
  - $P \wedge Q$
  - $\land$
  - $P \land Q$

- $P \wedge Q$ Extract Conclusion from
  - $P$
  - $\wedge$
  - $Q$

- $P \lor Q$ Extract from $\lor$
  - $P$
  - $\lor$
  - $Q$

- $P \land Q$ Extract Hypothesis from
  - $P$
  - $\land$
  - $Q$

- $P \lor Q$ Create $\lor$
  - $P$
  - $\lor$
  - $Q$

- $P \land Q$ Create $\land$
  - $P$
  - $\land$
  - $Q$

- $P \land Q$ Extract from $\land$
  - $P$
  - $\land$
  - $Q$
Example: Prove $(P \lor Q \rightarrow R) \land (R \rightarrow S \lor W) \rightarrow (Q \land \neg S \rightarrow W)$ is a theorem.

Note: By using the conditional proof law, this is equivalent to

$(P \lor Q \rightarrow R) \land (R \rightarrow S \lor W) \land (Q \land \neg S) \rightarrow W$

Conclusion with $\rightarrow$

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<td>$P \lor Q \rightarrow R$</td>
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<td>$R \rightarrow S \lor W$</td>
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<td>3.</td>
<td>$Q \land \neg S$</td>
<td>(hypothesis)</td>
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** Target: Show $W$ is true: **

4. $Q$ (3, simplification)
5. $P \lor Q$ (4, addition)
6. $R$ (1, 5 mp)
7. $S \lor W$ (2, 6 mp)
8. $\neg S$ (3, simplification)
9. $W$ (7, 8 disjunctive syllogism)

Example: Is the following argument valid (use propositional logic)?

Gary is intelligent or a good actor.
If Gary is intelligent, then he can count from 1 to 10.
Gary can only count from 1 to 3.
Therefore, Gary is a good actor.

i: “Gary is intelligent.”
a: “Gary is a good actor.”
c: “Gary can count from 1 to 10.”

Template:
(1) Identify all propositions and argument in the prompt
(2) Translate them in P-Logic
(3) Show the argument is valid

$(i \lor a) \land (i \rightarrow c) \land \neg c \rightarrow a$

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Step 1:  \( \neg c \)  
Step 2:  \( i \rightarrow c \)  
Step 3:  \( i \lor a \)  
Step 4:  \( \neg i \)  
Step 5:  \( a \)  

Another example:

If you listen to me, you will pass CS 230.

You passed CS 230.

Therefore, you have listened to me.

Is this argument valid?

No, the statement \((p \rightarrow q) \land q) \rightarrow p\) is not a tautology.

It is false if \(p\) is false and \(q\) is true.

You did not pass CS230 therefore you have not listened to me?

\((p \rightarrow q) \land \neg q \rightarrow \neg p\)

2.1 Introduction to Predicate Logic

- Recall a statement “\(x < 777\)” which is not a proposition. Let’s define a function \(P(x)\) indicating “\(x < 777\)”
- \(P\) is called **predicate** and \(x\) is the **variable**
- We can **instantiate** a predicate by giving a constant value to the variable, making a predicate to a proposition
- Truth value depends on value of variable
  - The truth value of \(P(500)\) is “True” (\(x=500\))
  - The truth value of \(P(1000)\) is “False” (\(x=1000\))
- Other Examples:
  
  \(\text{greater}(x, y)\) = which is true if \(x > y\); false otherwise
  
  \(\text{prime}(x)\) = which is true if \(x\) is a prime number, false otherwise
Syntax of Predicate Logic:

- A **predicate** is a function, in general, has the form

\[ P(x_1, x_2, \ldots, x_n) \]

which maps from \( x_1, x_2, \ldots, x_n \) to the truth values **true** and **false**.

where \( P \) is the name of the predicate,
\( x_i \) are variables or parameters
\( n \) is the degree of the predicate
\( = \# \) of variables

- **Proposition** is simply a declarative statement that is either true or false, has no variables involved.

- But **predicates can take variables**, and once we replace the variable by a constant (**instantiate**), it becomes a proposition. **Predicate instantiate \( \rightarrow \) Proposition!**

- Two quantifiers: (what for?)

  - Universal quantifier \( \forall \) (for all),
  - Existential quantifier \( \exists \) (exists),

A nonempty set \( U \) which is called universe or domain.
• $\forall x P(x)$ (For all $x$ $P(x)$ is true) $\quad U = \{a_1, \ldots, a_n\}$, $x \in U$
  - is true if $P(x)$ is true for every $x$ in $U$
  - otherwise, false.
  $$\forall x P(x) \iff P(a_1) \land P(a_2) \land \ldots \land P(a_{n-1}) \land P(a_n)$$

• $\exists x P(x)$ (There exists $x$ such that $P(x)$ is true)
  - is true if $P(x)$ is true for some/at least one $x$ in $U$
  - is false if $P(x)$ is false for every $x$ in $U$
  $$\exists x P(x) \iff P(a_1) \lor P(a_2) \lor \ldots \lor P(a_{n-1}) \lor P(a_n)$$

• $P(x)$ is a predicate.
• $\forall x P(x)$, $\exists x P(x)$: either true or false, so they are proposition

Example:

Let $U =$ all people

$S(x): x$ is a SFSU student.

$G(x): x$ is a genius.

$P(x): x$ is a SFSU professor.

(i) What does $\forall x (S(x) \to G(x))$ mean?
“For all $x$ in $U$, If $x$ is a SFSU student, then $x$ is a genius.” or “All SFSU students are geniuses.”

Note: $\forall x (S(x) \land G(x))$
means “all people are SFSU students and are genius”
• Example:

Let $U =$ all people
$S(x):$ $x$ is a SFSU student.
$G(x):$ $x$ is a genius.
$P(x):$ $x$ is a SFSU professor.

(ii) What does $\exists x \ (P(x) \land G(x))$ mean? 

“There is an $x$ in $U$ such that $x$ is a SFSU professor and $x$ is a genius.” or

“At least one SFSU professor is a genius.”

“Some SFSU professor is genius.”

Note: $\exists x \ (P(x) \rightarrow G(x))$
means “There is $x$ in $U$, if $x$ is a SFSU professor, then $x$ is genius”, This is true even for all who is not a SFSU professor!

• Example:

Let $U =$ all people
$S(x):$ $x$ is a SFSU student.
$G(x):$ $x$ is a genius.
$P(x):$ $x$ is a SFSU professor.

Let $U=$ all SFSU students
the statement “All SFSU students are geniuses” can be written as $\forall x \ G(x)$ instead of $\forall x \ (S(x) \rightarrow G(x))$
Another example:
Let \( U = \) the real numbers (\( \mathbb{R} \)).
What does \( \forall x \exists y \ (x + y = 320) \) mean?
“For every \( x \) there exists a \( y \) such that \( x + y = 320 \).”
Q: Is it true?
Q: Is it true for \( U = \) the natural numbers (\( \mathbb{N} \))?
Q: Counter example?

Note: \( \forall x \exists y \ (x + y = 320) = \forall x (\exists y \ (x + y = 320)) \)

Translate English sentences to predicates
Example:
\( U = \) all reports
\( A(x): x \) is authorized
\( T(x): x \) is trustworthy
\( F(x): x \) is false

Some unauthorized reports are false.
\( \exists x \ [\neg A(x) \land F(x)] \)

All authorized reports are trustworthy.
\( \forall x \ [A(x) \longrightarrow T(x)] \)

Some false reports are not trustworthy.
\( \exists x \ [F(x) \land \neg T(x)] \)
De Morgan’s law for quantifiers (They are interchangeable!):

- **Not all of...**  $\neg(\forall x \ P(x)) \iff \exists x \ (\neg P(x))$
- **None of...**  $\neg(\exists x \ P(x)) \iff \forall x \ (\neg P(x))$

Example: Not all roses are red

\[ \neg \forall x \ (Rose(x) \rightarrow Red(x)) \]  
\[ \iff \neg \forall x \ (\neg Rose(x) \vee Red(x)) \]  
\[ \iff \exists x \ (\neg Rose(x) \vee \neg Red(x)) \]  
\[ \iff \exists x \ (Rose(x) \land \neg Red(x)) \]  

Some roses are not red

Example: Nobody is perfect

\[ \neg \exists x \ (Person(x) \land Perfect(x)) \]  
\[ \iff \forall x \neg (Person(x) \land Perfect(x)) \]  
\[ \iff \forall x \neg Person(x) \lor \neg Perfect(x) \]  
\[ \iff \forall x Person(x) \rightarrow \neg Perfect(x) \]  

Everyone is imperfect