Formal Logic 3

• Study HW3! Due in one and half week (10/3)
• Last lecture: “How to prove, using propositional logic”
  – Normal forms
  – Deductive Proofs: Show that
    “Conditions $\rightarrow$ Conclusion” is a tautology
    by applying a series of Inference Rules such as Modus Ponens

• This lecture: Cont. of Logical Proofs
  – Deductive Proofs Cond.
  – Predicate Logic
Few Inferences rules (more in the text book)

\[ P \land Q \]

\[ P \lor Q \text{ disjunctive syllogism} \]

\[ \neg Q \]

\[ \neg P \text{ Modus Tollens} \]

\[ P \rightarrow Q \]

\[ P \lor Q \text{ addition} \]

\[ P \land Q \text{ simplification} \]

\[ P \rightarrow Q \text{ Modus Ponens} \]
Example: Prove \((P \lor Q \rightarrow R) \land (R \rightarrow S \lor W) \rightarrow (Q \land \neg S \rightarrow W)\) is a theorem.

Note: This is equivalent \((P \lor Q \rightarrow R) \land (R \rightarrow S \lor W) \land (Q \land \neg S) \rightarrow W\)

1. \(P \lor Q \rightarrow R\) (hypothesis)
2. \(R \rightarrow S \lor W\) (hypothesis)
3. \(Q \land \neg S\) (hypothesis)

**Target: Show \(W\) is true:**

4. \(Q\) (3, simplification)
5. \(P \lor Q\) (4, addition)
6. \(R\) (1, 5 mp)
7. \(S \lor W\) (2, 6 mp)
8. \(\neg S\) (3, simplification)
9. \(W\) (7, 8 disjunctive syllogism)
Example: Is the following argument valid (use propositional logic)?

Gary is intelligent or a good actor.
If Gary is intelligent, then he can count from 1 to 10.
Gary can only count from 1 to 3.
Therefore, Gary is a good actor.

i: “Gary is intelligent.”
a: “Gary is a good actor.”
c: “Gary can count from 1 to 10.”

\[(i \lor a) \land (i \rightarrow c) \land \neg c \rightarrow a\]
Step 1: \( \neg c \)  
Hypothesis
Step 2: \( i \rightarrow c \)  
Hypothesis
Step 3: \( i \lor a \)  
Hypothesis
Step 4: \( \neg i \)  
1 & 2, Modus tollens
Step 5: \( a \)  
3 & 4, Disjunctive Syllogism

Another example:

\[
P \quad q - \\
\text{If you listen to me, you will pass CS 230.} \\
\text{You passed CS 230.} \\
\text{Therefore, you have listened to me.}
\]

Is this argument valid?

No, the statement \( ((p \rightarrow q) \land q) \rightarrow p \) is not a tautology.
It is false if \( p \) is false and \( q \) is true.
You did not pass CS230 therefore you have not listened to me?
2.1 Introduction to Predicate Logic

• Recall a statement “x < 777” which is not a proposition. Define a function P(x) as “x < 777”
• P is called **predicate** and x is the variable
• We can **instantiate** a predicate by giving a constant value to the variable, making a predicate to a proposition
• Truth value depends on value of variable
  The truth value of P(500) is “True” (x=500)
  The truth value of P(1000) is “False” (x=1000)

• Other Examples:
  greater(x, y) = which is true if x > y; false otherwise
  prime(x) = which is true if x is a prime number, false otherwise
• A **predicate** is a function, in general, has the form

\[ P(x_1, x_2, ..., x_n) \]

which maps from \( x_1, x_2, ..., x_n \) to the values **true** and **false**.

where \( P \) is the name of the predicate, \( x_i \) are variables or parameters, and \( n \) is the degree of the predicate.
• **Proposition** is simply a declarative statement that is either true or false, has no variables involved.

• But **predicates can take variables**, and once we replace the variable by a constant (**instantiate**), it becomes a proposition.

• Two quantifiers: (what for?)

  - **Universal quantifier** \( \forall \) (for all),
  - **Existential quantifier** \( \exists \) (exists),

A nonempty set \( U \) which is called universe or domain.
\[ \forall x \ P(x) \quad \text{For all } x \ P(x) \text{ is true} \]
- is true if \( P(x) \) is true for every \( x \) in \( U \)
\[ \forall x \ P(x) = P(a_1) \land P(a_2) \land \cdots \land P(a_n) \]
- otherwise, false.

\[ \exists x \ P(x) \quad \text{There exists } x \text{ s.t. } P(x) \text{ is true} \]
- is true if \( P(x) \) is true for at least one \( x \) in \( U \)
- is false if \( P(x) \) is false for every \( x \) in \( U \)
\[ \exists x \ P(x) = P(a_1) \lor P(a_2) \lor \cdots \lor P(a_n) \]

- \( P(x) \) is a predicate.
- \( \forall x \ P(x) , \exists x \ P(x) : \) either true or false, so they are proposition
Example:

Let U = all people
S(x): x is a SFSU student.
G(x): x is a genius.
P(x): x is a SFSU professor.

(i) What does \( \forall x (S(x) \rightarrow G(x)) \) mean?

“For all x in U, If x is a SFSU student, then x is a genius.” or
“All SFSU students are geniuses.”

Note: \( \forall x (S(x) \land G(x)) \) means “all people are SFSU students and are genius”
Example:

Let $U = \text{all people}$
$S(x): x$ is a SFSU student.
$G(x): x$ is a genius.
$P(x): x$ is a SFSU professor.

(ii) What does $\exists x \ (P(x) \land G(x))$ mean?

“There is an $x$ in $U$ such that $x$ is a SFSU professor and $x$ is a genius.” or
“At least one SFSU professor is a genius.”
“Some SFSU professor is genius.”

Note: $\exists x \ (P(x) \rightarrow G(x))$
means “There is $x$ in $U$, if $x$ is a SFSU professor, then $x$ is genius”, This is true even when “x” is not a SFSU professor!
• **Example:**

Let $U = \text{all people}$

$S(x): x$ is a SFSU student.

$G(x): x$ is a genius.

$P(x): x$ is a SFSU professor.

Let $U = \text{all SFSU students}$

the statement “All SFSU students are geniuses” can be written as $\forall x \ G(x)$ instead of $\forall x \ (S(x) \rightarrow G(x))$
Another example:
Let $U = \mathbb{R}$.
What does $\forall x \exists y \ (x + y = 320)$ mean?
“For every $x$ there exists a $y$ such that $x + y = 320$.”

$x + y = 320 \Rightarrow y = 320 - x$.

Is it true?

Is it true for $U = \mathbb{N}$?

Counter example?

Note: $\forall x \exists y \ (x + y = 320) = \forall x (\exists y \ (x + y = 320))$
Translate English sentences to predicates
Example:

U = all reports
A(x): x is authorized
T(x): x is trustworthy
F(x): x is false

Some unauthorized reports are false.

\[ \exists x \left[ \neg A(x) \land F(x) \right] \]

All authorized reports are trustworthy.

\[ \forall x \left[ T(x) \rightarrow A(x) \right] \]

Some false reports are not trustworthy.

\[ \exists x \left[ \neg T(x) \land F(x) \right] \]
De Morgan’s law for quantifiers (They are interchangeable!):

\[
\neg (\forall x \ P(x)) \iff \exists x \ (\neg P(x)) \\
\neg (\exists x \ P(x)) \iff \forall x \ (\neg P(x))
\]

Example: Not all roses are red

\[
\neg \forall x \ (\text{Rose}(x) \rightarrow \text{Red}(x)) \\
\exists x \ (\text{Rose}(x) \land \neg \text{Red}(x))
\]

Example: Nobody is perfect

\[
\neg \exists x \ (\text{Person}(x) \land \text{Perfect}(x)) \\
\forall x \ (\text{Person}(x) \rightarrow \neg \text{Perfect}(x))
\]

Everyone is imperfect.