Formal Logic 2

- HW2 Due & HW3 Assignments will be online!
- HW3 is due in 1.5 weeks on Mar 2 Tuesday.
- Last lecture
  - Propositional Logic
  - Propositions, Statements, Connectives, Truth table, Formula
  - WFF Properties: Tautology, Contradiction, Validity, Satisfiability
  - Logical Equivalence
  - Tautology Equivalence Laws

- This lecture: Standard Procedure of Inferencing
  - Logical Equivalence Examples
  - Normal forms \( \leftarrow \) Standard
  - Deductive Proofs in Logic using Inference Rules

Example: \( P = \) Today is Monday, \( Q = \) I’ll go to London.
Today is Monday or I’ll go to London, but not both.
\( P \oplus Q \)
If and only if today is not Monday then I’ll go to London.
\( \neg P \leftrightarrow Q \)

\[
\begin{array}{c|c|c|c|c}
P & Q & \neg P & P \oplus Q & \neg P \leftrightarrow Q \\
\hline
T & T & F & F & F \\
T & F & F & T & T \\
F & T & T & T & T \\
F & F & T & F & F \\
\end{array}
\]

\( (P \oplus Q) \leftrightarrow (\neg P \leftrightarrow Q) \)

The propositions \( P \oplus Q \) and \( \neg P \leftrightarrow Q \) are logically equivalent, since they have the same truth values, or put it in another way, \( (P \oplus Q) \leftrightarrow (\neg P \leftrightarrow Q) \) is always true (tautology / valid formula).
• Example 1: Show that \((P \rightarrow Q)\) and \((-P \vee Q)\) are logically equivalent

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>(\neg P)</th>
<th>(P \rightarrow Q)</th>
<th>(-P \vee Q)</th>
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</thead>
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Truth Table shows that they have the same truth values thus they are logical equivalent.

• Example 2: Show that \((P \land Q) \rightarrow (P \lor Q)\) is a tautology

\[(P \land Q) \rightarrow (P \lor Q)\]
\[\iff \neg (P \land Q) \lor (P \lor Q)\] (by Rewriting Impl. law or example 1 above)
\[\iff (\neg P \lor \neg Q) \lor (P \lor Q)\] (by DeMorgan law)
\[\iff (\neg P \lor P) \lor (\neg Q \lor Q)\] (by Commutative and Associative laws)
\[\iff T \lor T\] (by Complement law)
\[\iff T\] (trivial)

Normal Form:

- Two special forms for formulas logically equivalent to a given formula: **Disjunctive normal form (DNF)** and **Conjunctive normal form (CNF)**.

  \[(\bigcirc \Lambda \bigstar \vee (\triangle \Lambda \bigstar) \vee (\bigcirc \vee \bigstar)\]

- **DNF**: a formula \(G\) of \(m\) variables being a disjunction \(x_1 \lor x_2 \lor \ldots \lor x_k\) of \(k \geq 0\) clauses/terms, where each \(x_i\) is a conjunction of \(m\) literals, i.e. \(x_i = (y_1 \land y_2 \land \ldots \land y_m)\)

- **CNF**: a formula \(G\) of \(m\) variables being a conjunction \(x_1 \land x_2 \land \ldots \land x_k\) of \(k \geq 0\) clauses, where each \(x_i\) is a disjunction of \(m\) literals, i.e. \(x_i = (y_1 \lor y_2 \lor \ldots \lor y_m)\)

- Theorem: Every formula is logically equivalent to a corresponding formula in DNF (and a formula in CNF) (in another word, every formula can be written in DNF and CNF)
Systematic way to find DNF and CNF:
Given $S = \neg ((P \rightarrow (Q \lor R)) \leftrightarrow (Q \rightarrow P))$

- Take each row where $S$ is “True”.
- DNF is $(\neg P \land Q \land \neg R) \lor (\neg P \land \neg Q \land \neg R) \lor (P \land \neg Q \land R) \lor (P \land Q \land R)$

Template for DNF:
(1) Make a Truth Table
(2) Find all interpretations that make $S$ true
(3) Construct a conjunctive clause for each interpretation above
(4) Combine all clauses by disjunction

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$S$</th>
<th>$\neg S$</th>
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</thead>
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- CNF for $S$ starts with DNF of $\neg S$, i.e. take each row where $S$ is “False” and form DNF. Then, use De Morgan’s law to convert $\neg (\neg S)$ in DNF to CNF

DNF for $\neg S$ is:
$(-P \land -Q \land -R) \lor (-P \land -Q \land -R) \lor (P \land -Q \land R) \lor (P \land Q \land R)$

Now $S = (\neg S)$:
$\neg (-P \land -Q \land -R) \land (-P \land -Q \land -R) \land (P \land -Q \land R) \lor (P \land Q \land R)$

Use DeMorgan’s Law: (push in 1st “$\neg$”)
$-(-P \land -Q \land -R) \land (-P \land -Q \land -R) \land (P \land -Q \land R) \land (P \land Q \land R)$

More DeMorgan’s Law to get CNF for $S$:
$(P \lor Q \lor -R) \land (P \lor Q \lor R) \land (-P \lor Q \lor -R) \land (-P \lor -Q \lor R)$
**Deductive Proof Method:**

Terminology:
- **axiom** is a basic assumption about mathematical structure that needs no proof, i.e. things known to be true (facts or proven theorems or tautologies)

A **theorem** is a statement that can be shown to be valid.

A **lemma** is a simple theorem used as an intermediate result in the proof of another theorem.

A **corollary** is a proposition that follows directly from a theorem that has been proved.

A **conjecture** is a statement whose truth value is unknown. Once it is proven, it becomes a theorem.

A theorem often has two parts in its *conditional statement*:

- **Conditions** (or hypotheses/premises) and a **Conclusion**

A correct (deductive) proof is to establish that

If all conditions are true, then the conclusion is true i.e., *(Conditions → Conclusion)* is valid (a tautology)

**Deduction**: a method which uses tautology laws and inference rules to prove a theorem, i.e. Often there are missing pieces (not easily understandable) between conditions and conclusion. Fill it by a sequence of applications of tautologies and inference rules

- Starting from conditions and axioms

- Generate a sequence of valid statements connected by proper rules of inference (new statements are generated from existing ones by these rules) to the conclusion
Logically valid inferences by using inference rules:

The general form of a rule of inference is:

\[ p_1 \quad p_2 \quad \ldots \quad p_n \quad \rightarrow \quad q \]

- if \( p_1 \) and \( p_2 \) and ... and \( p_n \) are all true, then \( q \) is true as well.
- Each rule is an established tautology of \( p_1 \land p_2 \land \ldots \land p_n \rightarrow q \)
- These rules of inference can be used in any mathematical argument and do not require any proof.

Modus ponens Latin name, means: method of assertion

\[ P \rightarrow (P \rightarrow Q) \quad \leftrightarrow \quad [P \land (P \rightarrow Q)] \rightarrow Q \]

\( P: \) I am hungry
\( Q: \) I eat

- An argument consists of one or more hypotheses and a conclusion. An argument is valid (theorem), if whenever all its hypotheses are true, its conclusion is also true.

Example: Use deduction to prove that \( [P \rightarrow (P \rightarrow Q)] \rightarrow (P \rightarrow Q) \) is a valid argument (or theorem).

Note: Using “Conditional proof” law, this is equivalent to

\[ \begin{align*}
\h1 & \quad [P \rightarrow (P \rightarrow Q)] \\
\h2 & \quad P \rightarrow Q \\
\text{Conc without } & \quad A \rightarrow (B \rightarrow C) \leftrightarrow A \land B \rightarrow C
\end{align*} \]

1. \( P \rightarrow (P \rightarrow Q) \) (hypothesis)
2. \( P \) (hypothesis)
3. \( P \rightarrow Q \) (1 & 2 modus ponens)
4. \( Q \) (2 & 3 modus ponens) \( \Box \)
Example: Prove that \( (\neg P \rightarrow \neg Q) \land (P \rightarrow S) \rightarrow (Q \rightarrow S) \) is theorem.

Note: By using the “Conditional proof” law, this is equivalent to \( (\neg P \rightarrow \neg Q) \land (P \rightarrow S) \land Q \rightarrow S \)

1. \( \neg P \rightarrow \neg Q \) (hypothesis)
2. \( P \rightarrow S \) (hypothesis)
3. \( Q \) (hypothesis)
4. \( \neg \neg Q \rightarrow \neg \neg P \) (contraposition, 1, 4 equivalent)
5. \( Q \rightarrow P \) (4, double negations)
6. \( P \) (3, 5 modus ponens)
7. \( S \) (2, 6 modus ponens) \( \square \)

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Important Inferences rules (more in the textbook)

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
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<tbody>
<tr>
<td>( P \rightarrow Q )</td>
<td>Modus Ponens</td>
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<tr>
<td>( \neg P \rightarrow \neg Q )</td>
<td>Conjunction</td>
</tr>
<tr>
<td>( P \lor Q \rightarrow \neg Q \rightarrow P )</td>
<td>Disjunctive Syllogism</td>
</tr>
<tr>
<td>( P \rightarrow Q )</td>
<td>Modus Tollens</td>
</tr>
<tr>
<td>( \neg Q \rightarrow \neg P )</td>
<td>Addition</td>
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</table>

- Extract Conclusion from \( \rightarrow \)
- Create \( \land \)
- Extract from \( \lor \)
- Extract Hypothesis from \( \rightarrow \)
- Create \( \lor \)
- Extract from \( \land \)
Example: Prove that \((A \rightarrow Q) \land (A \rightarrow L) \rightarrow [A \rightarrow (Q \land L)]\) is a theorem.

Note: By using the conditional proof law, this is equivalent to
\[
\frac{(A \rightarrow Q) \land (A \rightarrow L) \land A \rightarrow (Q \land L)}{\frac{Q}{L} \rightarrow (Q \land L) \text{ Conc}}
\]

1. \(A \rightarrow Q\) (hypothesis)
2. \(A \rightarrow L\) (hypothesis)
3. \(A\) (hypothesis)
4. \(Q\) (1, 3 modus ponens)
5. \(L\) (2, 3 modus ponens)
6. \(Q \land L\) (4, 5 conjunction inference rule) □

Example: Prove \((P \lor Q \rightarrow R) \land (R \rightarrow S \lor W) \rightarrow (Q \land \neg S \rightarrow W)\) is a theorem.

Note: By using the conditional proof law, This is equivalent to
\[
\frac{(P \lor Q \rightarrow R) \land (R \rightarrow S \lor W) \land (Q \land \neg S) \rightarrow W}{\text{Conc}}
\]

1. \(P \lor Q \rightarrow R\) (hypothesis)
2. \(R \rightarrow S \lor W\) (hypothesis)
3. \(Q \land \neg S\) (hypothesis)

** Target: Show \(W\) is true: **
4. \(Q\) (3, simplification)
5. \(P \lor Q\) (4, addition)
6. \(R\) (1, 5 mp)
7. \(S \lor W\) (2, 6 mp)
8. \(\neg S\) (3, simplification)
9. \(W\) (7,8 disjunctive syllogism) □