Formal Logic 2

- HW2 Due & HW3 Assignments will be online!
- HW3 is due in 1.5 weeks on 9/29.
- Last lecture
  - Propositional Logic
  - Propositions, Statements, Connectives, Truth table, Formula
  - WFF Properties: Tautology, Contradiction, Validity, Satisfiability
  - Logical Equivalence
  - Tautology Equivalence Laws

- This lecture: Standard Procedure of Inferencing
  - Logical Equivalence Examples
  - Normal forms ← Standard
  - Deductive Proofs in Logic using Inference Rules

Example: \( P = \) Today is Monday, \( Q = I’ll \) go to London. Today is Monday or I’ll go to London, but not both. \( P \oplus Q \)

If and only if today is not Monday then I’ll go to London. \( \neg P \leftrightarrow Q \)

\[
\begin{array}{cccccc}
P & Q & \neg P & P \lor Q & \neg P \leftrightarrow Q \\
T & T & F & T & F \\
T & F & F & T & T \\
F & T & T & T & T \\
F & F & T & F & F \\
\end{array}
\]

\((P \oplus Q) \leftrightarrow (\neg P \leftrightarrow Q)\)

The propositions \( P \oplus Q \) and \( \neg P \leftrightarrow Q \) are logically equivalent, since they have the same truth values, or put it in another way, \((P \oplus Q) \leftrightarrow (\neg P \leftrightarrow Q)\) is always true (tautology / valid formula).
Example 1: Show that \((P \rightarrow Q)\) and \((-P \lor Q)\) are logically equivalent

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(-P)</th>
<th>(P \rightarrow Q)</th>
<th>(-P \lor Q)</th>
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</thead>
<tbody>
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Truth Table shows that they have the same truth values thus they are logically equivalent.

Example 2: Show that \((P \land Q) \rightarrow (P \lor Q)\) is a tautology

\[
(P \land Q) \rightarrow (P \lor Q)
\]

\[
\iff \quad \neg(P \land Q) \lor (P \lor Q)
\]

(by Rewriting Impl. law or example 1 above)

\[
\iff \quad (\neg P \lor \neg Q) \lor (P \lor Q)
\]

(by DeMorgan law)

\[
\iff \quad (\neg P \lor \neg Q) \lor (Q \lor Q)
\]

(by Commutative and Associative laws)

\[
\iff \quad T \lor T
\]

(by Complement law)

\[
\iff \quad T
\]

(trivial)

Normal Form:

Two special forms for formulas logically equivalent to a given formula: **Disjunctive normal form (DNF)** and **Conjunctive normal form (CNF)**.

- **DNF**: a formula \(G\) of \(m\) variables being a disjunction \(x_1 \lor x_2 \lor \ldots \lor x_k\) of \(k \geq 0\) clauses/terms, where each \(x_i\) is a conjunction of \(m\) literals, i.e. \(x_i = (y_1 \land y_2 \land \ldots \land y_m)\)

  = propositional symbol

  \[(\bigvee \bigwedge \bigvee \bigwedge) \lor (\bigwedge \bigvee \bigwedge) \lor (\bigwedge \bigwedge)\]

- **CNF**: a formula \(G\) of \(m\) variables being a conjunction \(x_1 \land x_2 \land \ldots \land x_k\) of \(k \geq 0\) clauses, where each \(x_i\) is a disjunction of \(m\) literals, i.e. \(x_i = (y_1 \lor y_2 \lor \ldots \lor y_m)\)

  = propositional symbol

  \[(\bigvee \bigvee \bigvee \bigvee) \land (\bigvee \bigvee \bigvee) \land (\bigvee \bigvee)\]

Theorem: Every formula is logically equivalent to a corresponding formula in DNF (and a formula in CNF) (in another word, every formula can be written in DNF and CNF)
Systematic way to find DNF and CNF:
Given \( S = \neg ((P \rightarrow (Q \lor R)) \leftrightarrow (Q \rightarrow P)) \)

- Take each row where \( S \) is “True”.
- DNF is \( \neg P \land Q \land \neg R \lor \neg P \land \neg Q \land \neg R \lor P \land \neg Q \land R \lor P \land Q \land R \lor P \land \neg Q \land \neg R \)
- CNF for \( S \) starts with DNF of \( \neg S \), i.e. take each row where \( S \) is “False” and form DNF. Then, use De Morgan’s law to convert \( \neg (\neg S) \) in DNF to CNF.

Template for DNF:
(1) Make a Truth Table
(2) Find all interpretations that make \( S \) true
(3) Construct a conjunctive clause for each interpretation above
(4) Combine all clauses by disjunction

Check this!

DNF for \( \neg S \) is:
\[
(\neg P \land \neg Q \land \neg R) \lor (\neg P \land \neg Q \land \neg R) \lor (P \land \neg Q \land R) \lor (P \land Q \land R) \lor (P \land \neg Q \land \neg R)
\]

Now \( S = \neg (\neg S) : \)
\[
\neg [ (\neg P \land \neg Q \land \neg R) \lor (\neg P \land \neg Q \land \neg R) \lor (P \land \neg Q \land R) \lor (P \land Q \land \neg R) ]
\]

Use DeMorgan’s Law: (push in 1st “ \( \neg \)”)
\[
\neg (P \land \neg Q \land R) \land (P \land \neg Q \land \neg R) \land (P \land \neg Q \land R) \land (P \land Q \land \neg R)
\]

More DeMorgan’s Law to get CNF for \( S \):  
\[
(P \lor Q \lor \neg R) \land (P \lor Q \lor R) \land (\neg P \lor Q \lor \neg R) \land (\neg P \lor Q \lor R)
\]
**Deductive Proof Method:**

**Terminology:**
- **axiom** is a basic assumption about mathematical structure that needs no proof, i.e. things known to be true (facts or proven theorems or tautologies)

A **theorem** is a statement that can be shown to be valid.

A **lemma** is a simple theorem used as an intermediate result in the proof of another theorem.

A **corollary** is a proposition that follows directly from a theorem that has been proved.

A **conjecture** is a statement whose truth value is unknown. Once it is proven, it becomes a theorem.

A theorem often has two parts in its *conditional statement* **Conditions** (or hypotheses/premises) and a **Conclusion**

A correct (deductive) proof is to establish that If all **conditions** are true, then the **conclusion** is true i.e., (Conditions $\rightarrow$ Conclusion) is valid (a tautology)

**Deduction**: a method which uses tautology laws and inference rules to prove a theorem, i.e. Often there are missing pieces (not easily understandable) between conditions and conclusion. Fill it by a sequence of applications of tautologies and inference rules

- Starting from **conditions** and **axioms**

- Generate a sequence of valid statements connected by proper rules of inference (new statements are generated from existing ones by these rules) to the conclusion
Logically valid inferences by using inference rules:

The general form of a rule of inference is:

\[
\begin{align*}
p_1 \\
p_2 \\
. \\
. \\
p_n \\
\hline
q
\end{align*}
\]

- if \(p_1\) and \(p_2\) and ... and \(p_n\) are all true, then \(q\) is true as well.
- Each rule is an established tautology of \(p_1 \land p_2 \land \ldots \land p_n \to q\)
- These rules of inference can be used in any mathematical argument and do not require any proof.

Modus ponens Latin name, means: method of assertion

\[
\begin{align*}
P & \to Q \\
Q & \to Q
\end{align*}
\]

- An argument consists of one or more hypotheses and a conclusion. An argument is valid (theorem), if whenever all its hypotheses are true, its conclusion is also true.

Example: Use deduction to prove that \([P \to (P \to Q)] \to (P \to Q)\) is a valid argument (or theorem).

Note: Using “Conditional proof” law, this is equivalent to

\[
\begin{align*}
[(P \to (P \to Q)) \land \frac{P}{P} \to Q] & \quad (h1) \\
\frac{P \to (P \to Q)}{P} & \quad (h2) \\
\frac{Q}{Q} & \quad \text{Conc without } \to
\end{align*}
\]

| 1. \(P \to (P \to Q)\) | (hypothesis) |
| 2. \(P\) | (hypothesis) |
| 3. \(P \to Q\) | (1 & 2 modus ponens) |
| 4. \(Q\) | (2 & 3 modus ponens) □ |
• Example: Prove that \((\neg P \rightarrow \neg Q) \land (P \rightarrow S) \rightarrow (Q \rightarrow S)\) is theorem.

\[
A \rightarrow (B \rightarrow C) \iff A \land B \rightarrow C
\]

Note: By using the “Conditional proof” law, this is equivalent to

\[
(\neg P \rightarrow \neg Q) \land (P \rightarrow S) \land Q \rightarrow S
\]

1. \(\neg P \rightarrow \neg Q\) (hypothesis)
2. \(P \rightarrow S\) (hypothesis)
3. \(Q\) (hypothesis)
4. \(\neg Q \rightarrow \neg P\) (contraposition, 1, 4 equivalent)
5. \(Q \rightarrow P\) (4, double negations)
6. \(P\) (3, 5 modus ponens)
7. \(S\) (2, 6 modus ponens) \(\Box\)

Important Inferences rules (more in the textbook)

- \(P \rightarrow Q\) : Modus Ponens
  - Extract Conclusion from \(\rightarrow\)

- \(P \land Q\) : Conjunction
  - Create \(\land\)

- \(P \lor Q\) : Disjunctive Syllogism
  - Extract from \(\lor\)

- \(P \rightarrow Q\) : Modus Tollens
  - Extract Hypothesis from \(\rightarrow\)

- \(P \land Q\) : Addition
  - Create \(\lor\)

- \(P \lor Q\) : Simplification
  - Extract from \(\land\)
• Example: Prove that \((A \rightarrow Q) \land (A \rightarrow L) \rightarrow [A \rightarrow (Q \land L)]\)

is a theorem

Note: By using the conditional proof law, this is equivalent to

\[(A \rightarrow Q) \land (A \rightarrow L) \land A \rightarrow (Q \land L)\]

\hline
1. \(A \rightarrow Q\) (hypothesis)
2. \(A \rightarrow L\) (hypothesis)
3. \(A\) (hypothesis)
4. \(Q\) (1, 3 modus ponens)
5. \(L\) (2, 3 modus ponens)
6. \(Q \land L\) (4, 5 conjunction inference rule)

\hline

• Example: Prove \((P \lor Q \rightarrow R) \land (R \rightarrow S \lor W) \rightarrow (Q \land \neg S \rightarrow W)\)

is a theorem

Note: By using the conditional proof law, this is equivalent to

\[(P \lor Q \rightarrow R) \land (R \rightarrow S \lor W) \land (Q \land \neg S) \rightarrow W\]

\hline
1. \(P \lor Q \rightarrow R\) (hypothesis)
2. \(R \rightarrow S \lor W\) (hypothesis)
3. \(Q \land \neg S\) (hypothesis)
** Target: Show \(W\) is true: **
4. \(Q\) (3, simplification)
5. \(P \lor Q\) (4, addition)
6. \(R\) (1, 5 mp)
7. \(S \lor W\) (2, 6 mp)
8. \(\neg S\) (3, simplification)
9. \(W\) (7, 8 disjunctive syllogism)

Think Backward!