Overview

• HW2 due in one week. Work on it!

• Last lectures
  – Proof techniques: Direct, Exhaustive, Counterexample
  – Proof by Contraposition
  – Proof by Contradiction

• Today’s lectures: Completing Proof techniques!
  – Proof Exercise by Various Methods
  – Mathematical Induction
  – Strong form of mathematical induction

Chapter 1.2 : Proof Techniques Cond.
Mathematical Induction:

Proof by induction: to prove that the property \( p(n) \) is true for all possible value of \( n \).

Idea: like playing the domino game.

Suppose dominos are placed correctly, then hitting the 1\(^{st} \) domino, when it falls, we know the rest of them will also fall.

(\( \because \) In general, when the \( k^{th} \) one falls, it implies the \( (k+1)^{th} \) falls. Since \( k \) is any arbitrary number, \( \therefore \) actually every domino falls.)

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Template of Mathematical Induction:

Step1: (inductive base) or IB:  \( p(n) \) \( \forall \in \mathbb{N} \)

Show that \( p(n_0) \) is true. Choose an \( n_0 \in \mathbb{N} \) that is appropriate to the problem.

Note: \( n_0 \) is usually a small number 0 or 1 unless a range is specified.

Step 2: (inductive hypothesis) or IH:

Suppose \( p(k) \) is true for any \( k \geq n_0 \)

Step3: (inductive step) or IS:

Show that \( p(k + 1) \) is true (given \( p(k) \) is true) for all natural numbers \( k \) such that \( k \geq n_0 \)

(If \( p(k) \) is true then \( p(k+1) \) is also true; \( p(k) \rightarrow p(k+1) \) is true; if a domino falls, the next domino also falls)
Example 1: Show $1+3+5+\ldots+(2n-1) = n^2$, for all $n \geq 1$

IB: when $n_o = 1$, $LHS = 1$, $RHS = 1^2 = 1 \Rightarrow LHS = RHS$

IH: Suppose $1+3+5+\ldots+(2k-1) = k^2$ for $n = k \geq n_o$

IS: Show $1+3+5+\ldots+(2k-1) + [2(k+1)-1] = (k+1)^2$

\[
\begin{align*}
\therefore \quad LHS' &= 1+3+5+\ldots+(2k-1) + (2k+1) \\
&= k^2 + 2k + 1 \quad \text{(by IH)}
\end{align*}
\]

\[
\begin{align*}
&\text{IH} \quad \therefore \quad LHS = RHS, \\
&\therefore \quad LHS' = RHS' \quad \text{(by IH)}
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow \quad \text{LHS} = \text{RHS}, \\
&\Rightarrow \quad \text{LHS}' = \text{RHS}'.
\end{align*}
\]

Example 2: Show $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$, for all $n \geq 1$

IB: when $n_o = 1$, $LHS = 1$, $RHS = \frac{1(1+1)}{2} \Rightarrow LHS = RHS$

IH: Assume $1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$ for $n = k \geq n_o$

IS: Show $1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

\[
\begin{align*}
\therefore \quad LHS' &= \frac{(k+1)(k+2)}{2} \\
&= RHS'.
\end{align*}
\]
Example 3: Show \(1 + 2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 1\), for all \(n \geq 1\)

**IB:** when \(n_0 = 1\), \(LHS = 1 + 2 = 3, \ RHS = 2^2 - 1 = 3 \rightarrow L = R\)

**IH:** Suppose \(1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1\) for \(n = k \geq n_0\)

**IS:** Show \(1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1\) \((n = k + 1)\)

\[
LHS' = 1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} = \frac{k(k+1)}{2} + (k+1)
\]

(by IH)

\[
= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = RHS'
\]

\(\therefore x = \frac{k(k+1)}{2}\)
Example 4: Show $2^{2n} - 1$ is divisible by 3, for all $n \geq 1$

IB: When $n_o = 1$, $2^2 - 1 = 3$ :: divisible by 3

IH: Assume $2^{2k} - 1$ is divisible by 3
i.e. $2^{2k} - 1 = 3m$ for some integer $m$ for $n = k \geq n_o$

IS: Show $2^{2(k+1)} - 1$ is divisible by 3.

$2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^2 \cdot 2^{2k} - 1$

$= 2^2(3m+1) - 1$ (:: $2^{2k} - 1 = 3m$

$2^{2k} = 3m + 1$, by IH)

$= 12m + 3$

$= 3(4m + 1)$ where $4m + 1$ is an integer

:: is divisible by 3 □

Example 5 (Fibonacci number property) Show that:

\[ F_1 + F_3 + \ldots + F_{2n-1} = F_{2n} - 1, \text{ for all } n \geq 1. \]

Note: $F_o = F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for all $k > 2$

IB: When $n_o = 1$, LHS = $F_1 = 1$, RHS = $F_2 - 1 = 1$ :: L=R

IH: Suppose $F_1 + F_3 + \ldots + F_{2k-1} = F_{2k} - 1$, for $n=k \geq n_o$

IS: Show $F_1 + F_3 + \ldots + F_{2(k+1)-1} = F_{2(k+1)} - 1$

\[
\text{LHS}' = F_1 + F_3 + \ldots + F_{2k-1} + F_{2k+1} = F_{2k} - 1 + F_{2k+1} \quad \text{(by IH)}
\]

\[
= F_{2k+2} - 1 \quad \text{(by Fibonacci number formula)}
\]

\[
= F_{2(k+1)} - 1
\]

= RHS' □
Example 6 (Cardinality of a power set): For any set $X$ with $|X|=n$, $|P(X)| = 2^n$, for all $n \geq 0$.

IB: When $n = 0$, $LHS = |P(\emptyset)| = 1$, $RHS = 2^0 = 1 \therefore L = R$.

IH: Assume $|X|=k$, $|P(X)| = 2^k$ for $n = k \geq 0$.

IS: Show $|X|=k+1$, $|P(X)| = 2^{k+1}$.

Lemma: Let $A$ be any set and let $b \not\in A$. If $|P(A)|=m$, then $|P(A \cup \{b\})|=2m$.

Pick any element $b$ from $X$ in IS, $|X-\{b\}| = k$.

$\rightarrow$ $|P(X-\{b\})| = 2^k$ (by IH)

$\rightarrow$ $|P(X)| = 2 \times 2^k$ (by Lemma) = $2^{k+1}$ $\blacksquare$

In class exercise

Use induction to prove that:

$2^n < n!$, for all $n \geq 4$
Strong Form of Mathematical Induction (template):

Step 1: *(inductive base)* or IB is to show that $p(n_0)$ is true. Choose an $n_0 \in \mathbb{N}$ appropriate to the problem.

Step 2: *(inductive hypothesis)* or IH:

Suppose $p(m)$ is true for all $m : k \geq m \geq n_0$

Step 3: *(inductive step)* or IS is to show that $p(k) \rightarrow p(k + 1)$, for all $k$ such that $k \geq n_0$

Regular: $P(k) \rightarrow P(k+1)$

Strong Form: $P(k) & P(k-1) & P(k-2) & \ldots & P(n_0) \rightarrow P(k+1)$

Example: Given $a_0 = 0$, $a_1 = 2$ and $a_n = 4(a_{n-1} - a_{n-2})$ for all $n \geq 2$. Show that: $a_n = n \times 2^n$ for all $n$

IB: When $n = 0$, $LHS = a_0 = 0$, $RHS = 0 \times 2^0 = 0$

$n = 1$, $LHS = a_1 = 2$, $RHS = 1 \times 2^1 = 2$

$n = 2$, $LHS = a_2 = 8$, $RHS = 2 \times 2^2 = 8$

IH: Assume $a_k = k \times 2^k$ for all indices below $k$

IS: Show $a_{k+1}^{LHS'} = (k+1) \times 2^{k+1}$

$LHS' = 4(a_k - a_{k-1}) = 4(k \times 2^k - (k-1) \times 2^{k-1})$ (by IH)

$= 4(2k \times 2^{k-1} - k \times 2^{k-1} + 2^{k-1}) = 4(k+1)(2^{k-1})$

$= (k+1) \times 2^{k+1} = RHS'$ □