Overview

• HW#1 Due & HW2 Assignment online soon!
• HW2 is due on 1.5 week on Sep 16 Thr!
• Last lecture: Sets Completed (keep coming back...)
  – Proof by cases, Proof by using existing rules
  – Basic counting theorems
  – Principle of inclusion and exclusion
  – Counting exercises

• Today’s lecture: General Proof Techniques
  – Proof Techniques Review
  – Six General Ones
  – Direct Proof
  – Proof by Exhaustion
  – Proof by Counter Example
  – Proof by Contraposition
  – Proof by Contradiction

Chapter 1.2 : Proof Techniques
Review some set proof templates:

- \( x \in A \): show that \( x \) has all membership properties of \( A \)
- \( A \subseteq B \): show that every element of \( A \) is also in \( B \).
- \( A \subset B \): show \( A \subseteq B \) and also some element \( x \) of \( B \) is not in \( A \)
- \( A = B \): show that \( A \subseteq B \) and \( B \subseteq A \)
- \( A \neq B \): show that \( A \not\subseteq B \) or \( B \not\subseteq A \) by showing some element \( x \) of \( A \) or \( B \) is not in \( B \) or \( A \)
- \( A \rightarrow B \): suppose \( A \) is true then derive \( B \): “if \( A \), then \( B \)”
- \( A \leftrightarrow B \): show that \( A \Rightarrow B \) and \( B \Rightarrow A \)

Proof by Cases: Make Membership Tables

Proof by Using Existing Rules: Deductive Proof with Set Identities

Six general proof techniques:

1) **Exhaustive Proof**: (to prove \( P \) is true), "Proof by Case"
   Show that all possible cases for \( P \) are true, (only for finite cases)

2) **Direct Proof**: to prove \( P \rightarrow Q \) is true (if \( P \) is true, then \( Q \) is true),
   Show that, suppose \( P \) is true, then **deduce** \( Q \). "Proof by Existing Rule"
   (deductive)

3) **Contraposition**: to prove \( P \rightarrow Q \) is true
   Show \( \neg Q \rightarrow \neg P \) (\( \neg Q \) implies \( \neg P \))
   (indirect proof)

4) **Contradiction**: to prove \( P \rightarrow Q \) is true,
   Show \( P \) and \( \neg Q \) \rightarrow (contradiction):
   Assume both the hypothesis (\( P \)) and the negation of the conclusion (\( \neg Q \)) are true, then try to deduce some contradiction from this assumption.
5) **Counterexample**: to disprove something

6) **Induction**: to prove that $P(n)$ is true for all $n$, Use the principle of mathematical induction:

- **Base case**: $P(1)$ or $P(0)$ is true
- **For all** $k$, $[ P(k) \text{ true} \rightarrow P(k+1) \text{ true} ]$
- **Conclusion**: $P(n) \text{ true, } \forall n$

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**Proof by Exhaustion**: "Proof by case"

Example: Show that $n! < 2^n$ for any positive integer $n \leq 3$ (U?)

Proof:

List all possible cases:

- $n=1$, $1! < 2^1 \rightarrow 1 < 2$ (true)
- $n=2$, $2! < 2^2 \rightarrow 2 < 4$ (true)
- $n=3$, $3! < 2^3 \rightarrow 6 < 8$ (true)

□ / Q.E.D.
Example: if an integer between 5 and 15 is divisible by 6, then it is also divisible by 3

Proof: \[ \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\} \]

List all possible cases:
- \( n=6 \) is divisible by 6 and is divisible by 3
- \( n=12 \) is divisible by 6 and is divisible by 3
- All other \( n \) values are not divisible by 6

\[ \square \]

Note: If the above problem is for all integers, then we cannot use exhaustive proof

\[ \Rightarrow \mathbb{Z}, |\mathbb{Z}| = \infty \]

**Direct Proof: (deductive)**

Example: For all \( x \), if \( x \) is divisible by 6 then \( x \) is divisible by 3

Proof:

- if \( x \) is divisible by 6
  - \( x = k \times 6 \), for some integer \( k \)
  - \( x = k \times 2 \times 3 \)
  - \( x = (k \times 2) \times 3 \)
  - \( x = k' \times 3 \), where \( k' = k \times 2 \)
- since \( k' \) is integer, \( x \) is divisible by 3

\[ \square \]
Show \( P \Rightarrow Q \): \( P \): \( x \) is even and \( y \) is even, \( Q \): \( xy \) is even

Example: Show that the product of two even integers is even.
Proof: Let \( x = 2m, y = 2n \) for some integer \( m, n \)
then \( xy = (2m)(2n) = 2(2mn) \), which is even
\[ \therefore 2mn \text{ is integer. } \square \]

Show \( P \Rightarrow Q \): \( P \): \( x \) is odd and \( y \) is odd, \( Q \): \( x+y \) is even
Example: Show that the sum of two odd integers is even
Proof: Let \( x = 2m + 1, y = 2n + 1 \) for some integer \( m,n \)
then \( x + y = 2m + 2n + 2 = 2(m+n+1) \),
where \( m+n+1 \) is an integer
\[ \therefore x+y \text{ is even. } \square \]
**Proof by Counterexample:**

Proving $P$ to be false (disproof) is much easier than proving $P$ to be true (proof)!

PROOF: must show all cases are true

DISPROOF: **showing only one case that is not true suffices**!

There are two (three) types of questions in proof
1) Prove (disprove) a statement $P$ (Show that $P$ is true (false)).
2) Is a statement $P$ true? (Prove or disprove $P$/Determine whether $P$ is true or not)
3) Prove (disprove) a statement $P$ by using $X$ technique

The second question requires you to see if the statement $P$ is true or not. So you must consider both cases of $P$ is true and $P$ is false.

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**Examples for Proof by Counterexample:**

Example: Disprove that every integer less than 10 is bigger than 5.

To disprove (or prove the statement is not true), find a counterexample,

Let $n = 4 < 10$, but $n$ is not $> 5$. □

Example: Is the sum of any three consecutive integers even?

To disprove the statement, give a counterexample: $2+3+4=9$ □
**Proof by Contraposition:**

Example: Prove that: If the square of an integer is odd, then the integer must be odd.

**Template:** $P \implies Q \iff \neg Q \implies \neg P$

Prove: if $n^2$ is odd, then $n$ is odd (initial statement)

Prove: if $n$ is not odd, then $n^2$ is not odd (contraposition)

i.e. Prove: $n$ is even $\implies n^2$ is even

Let $n = 2m$ for some integer $m$

$\implies n^2 = n \times n = 2m \times 2m = 2(2m^2)$

$\implies$ since $2m^2$ is integer, $n^2$ is even. $\square$

**Example:** Show that $xy$ is odd if and only if both $x$ and $y$ are odd.

Proof:

$(\iff)$ if $x$ and $y$ are odd, then $xy$ is odd.

By direct proof:

Let $x = 2m+1$, $y=2n+1$ for some $m$, $n \in \mathbb{Z}$

$\implies xy = (2m+1)(2n+1) = 4mn + 2m + 2n + 1$

$\implies = 2(2mn + m + n) + 1$

$\implies$ since $2mn + m + n$ is an integer, $xy$ is odd.
If $xy$ is odd then $x$ and $y$ are odd.

**By contraposition**: if $x$ is not odd or $y$ is not odd, then $xy$ is not odd.

i.e. if $x$ even or $y$ even, then $xy$ even

**Exhaustive Proof**

**case 1** $x$ even, $y$ odd: Let $x = 2m$, $y = 2n+1$

$$xy = 2(2mn + m),$$

which is even $\therefore 2mn+m \in \mathbb{Z}$

**case 2** $x$ odd, $y$ even: similar to case 1.

**case 3** $x$ even, $y$ even: Let $x = 2m$, $y = 2n$

$$xy = 2(2mn),$$

which is even $\therefore 2mn \in \mathbb{Z}$ □

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**Proof by Contradiction**

- Prove/Show $P \rightarrow Q$ by contradiction method
- Is equivalent to show that $(P \land Q')$ deduces to a contradiction (violation of assumption)
- Logical proof of contradiction:
  - Let $x' = (P \rightarrow Q)' = P \land Q'$, we assume $x'$ and derive a contradiction $y'$, i.e. $x' \rightarrow y'$
  - Where $y'$ is false, i.e. $y$ is true (or axiom)
  - By *modus tollens*: $(x' \rightarrow y') \land y \rightarrow x$
  - Therefore, conclude $x = P \rightarrow Q$ is true □
Proof by Contradiction:

Example: If a number added to itself gives itself, then the number is 0, i.e. if \( x + x = x \), then \( x = 0 \)

Proof:
Assume \( x + x = x \) and \( x \neq 0 \)

\[ \rightarrow 2x = x \text{ and } x \neq 0 \]

\[ \rightarrow 2 = 1 \], which is a contradiction

\[ \therefore \text{ the assumption must be wrong} \]

\[ \therefore \text{ if } x + x = x \text{, then } x = 0 \quad \square \]

Example: Prove that if \( x^2 + 2x - 3 = 0 \), then \( x \neq 2 \)

1. by contradiction: \( P \) and \( \neg Q \rightarrow \rightarrow \text{ Contradiction} \)

Suppose \( x^2 + 2x - 3 = 0 \) and \( x = 2 \),

\[ \rightarrow 4 + 4 - 3 = 0 \text{ or } 5 = 0 \], which is a contradiction. \( \square \)

2. by direct proof:

\[ P \rightarrow P' \rightarrow \rightarrow Q \]

if \( x^2 + 2x - 3 = 0 \rightarrow (x + 3)(x - 1) = 0 \)

\[ \rightarrow x = -3 \text{ or } x = 1 \rightarrow x \neq 2 \quad \square \]

3. by contraposition: \( Q' \rightarrow \rightarrow P' \)

show that if \( x = 2 \), then \( x^2 + 2x - 3 \neq 0 \)

\[ \rightarrow x^2 + 2x - 3 = 5 \neq 0 \quad \square \]
In class exercises

Show that if $3n + 2$ is odd, then $n$ is odd.

a) Proof by contradiction

b) Proof by contraposition

c) Direct proof