Overview

• Study for HW#1 (Due next Tuesday before the class!)

• Last lecture
  – Proof templates for sets
  – Set operations: Union, Intersection, Difference, Sym-Diff
  – Cartesian Product

• Today’s lecture: Completing Lectures on Sets!
  – More Proof Templates
    • Proof by cases,
    • Proof by using existing rules
    • Review of Set identities: All *true* rules for sets that you can use for
      making your proof!
  – Basic counting theorem
  – Principle of inclusion and exclusion
  – Counting exercises

Chapter 1.1 : Set Theory

Cond.
More proof templates (will show example later):

- **Proof by cases**: List all possible cases/situations. For each case, prove the conclusion separately. (See example \( x \in A, A \subseteq B \))
- **Proof by using existing rules**: derive the desired conclusion from the assumption only by using the set identities.
- **Disproof by counterexample**: Find an \( x \) that can be proved to demonstrate that the conclusion is false. (See example \( A \neq B \))
Proof by using existing rules:

• **AKA: Deductive Proof:** Use the set identities (valid steps) to argue “if $A$, then $B$” is true.

• Starting from a statement that you know is true.

• Revise the statement by using the identity rules successively, until you derive the form of statement that you want to prove.

• Each step provides a true statement if you only use proper identities! (One mistake makes the entire proof wrong!)

• **HINT:** think backward! $\leftarrow$ long chain...

Set Identities (rules/properties)

1. Identity laws $A \cup \emptyset = A$, $A \cap U = A$
2. Domination laws $A \cup U = U$, $A \cap \emptyset = \emptyset$
3. Idempotent laws $A \cup A = A$, $A \cap A = A$
4. Complementation law $(\bar{A}) = (A')' = A$
5. Commutative laws $A \cup B = B \cup A$, $A \cap B = B \cap A$
6. Associative laws $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$
7. More properties: $A \subseteq A \cup B$, $B \subseteq A \cup B$, $A \cap B \subseteq A$, $A \cap B \subseteq B$, $A \Rightarrow A \subseteq A \cup B$, $A \cap B \Rightarrow A \cap B \subseteq A \cap B \subseteq B$
8. Distributive laws

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

9. De Morgan's laws

\[ \overline{A \cup B} = \overline{A} \cap \overline{B} \]
\[ \overline{A \cap B} = \overline{A} \cup \overline{B} \]

Note: \( \overline{A} = A^c = A' \)

10. Absorption laws

\[ A \cup (A \cap B) = A \]
\[ A \cap (A \cup B) = A \]

11. Complement laws

\[ A \cup \overline{A} = U \]
\[ A \cap \overline{A} = \emptyset \]

Note: You can prove all above identities. You may use basic definitions and existing identities to prove new set relations and properties!
(i) Suppose \( x \in A \cup (B \cap C) \). It is necessary to show that \( x \in (A \cup B) \cap (A \cup C) \) also.

Since \( x \in A \cup (B \cap C) \), \( x \in A \) or \( x \in (B \cap C) \) (by \( \cup \) definition)

Note: There are 2 cases to prove, we use proof by cases below.

Case I, \( x \in A \), then,
\[
\begin{align*}
x &\in (A \cup B) \text{ and } x \in (A \cup C) \quad \text{(by Identity \#7 or text theorem 1(b))} \\
\text{It follows } x \in (A \cup B) \cap (A \cup C) \quad \text{(by } \cap \text{ definition)}
\end{align*}
\]

Case II, \( x \in (B \cap C) \), then \( x \in B \) and \( x \in C \) (by \( \cap \) definition)
\[
\begin{align*}
x &\in (A \cup B) \text{ and } x \in (A \cup C) \quad \text{(by Identity \#7 or text theorem 1(b))} \\
\text{Therefore, } x \in (A \cup B) \cap (A \cup C) \quad \text{(by } \cap \text{ definition)}
\end{align*}
\]

(ii) Suppose \( x \in (A \cup B) \cap (A \cup C) \). It is necessary to show that \( x \in A \cup (B \cap C) \).

Since \( x \in (A \cup B) \cap (A \cup C) \), then \( x \in (A \cup B) \) and \( x \in (A \cup C) \) (by \( \cap \) definition)

This means \( (x \in A \) or \( x \in B) \) and \( (x \in A \) or \( x \in C) \) (by \( \cup \) definition)

Note : There are 4 possible cases

Case I : if \( x \in A \) and \( x \in A \)
then \( x \in A \)
then \( x \in A \cup (B \cap C) \) (by Identity \#7 or text theorem 1(b))

Case II : if \( x \in A \) and \( x \in C \)
then \( x \in A \cap C \) (by \( \cap \) definition)
then \( x \in A \) (by Identity \#7 or text theorem 2(b))
It follows \( x \in A \cup (B \cap C) \) (by Identity \#7 or text theorem 1(b))

Case III, if \( x \in B \) and \( x \in A \),
Analogous to Case II

Case IV, if \( x \in B \) and \( x \in C \),
then \( x \in B \cap C \) (by \( \cap \) definition)
It follows \( x \in A \cup (B \cap C) \) (by Identity \#7 or text theorem 1(b))
A-B, A\cap B, B-A are mutually exclusive/disjoint sets.

i.e. if \( x \in A-B \), then \( x \notin B \), therefore \( x \notin B-A \) and \( x \notin A\cap B \).

\[ U \]: universal set

\[
\left| A \cup B \right| = \left| A \right| + \left| B \right| - \left| A \cap B \right|
\]

\[
= \left| A - B \right| + \left| A \cap B \right| + \left| B - A \right|
\]

\[
\left| A - B \right| = \left| A \right| - \left| A \cap B \right|
\]

\[
\left| B - A \right| = \left| B \right| - \left| A \cap B \right|
\]

\[ \therefore \] e.g. \( |A \cap B| \) can be computed in several ways depends on the information given.

Other Counting Theorems:

Let \( A \) and \( B \) be subsets of a finite universal set \( U \)

(a) Let \( B \subseteq A \), then:

\[
\left| B \right| \leq \left| A \right|
\]

\[
\left| A - B \right| = \left| A \right| - \left| B \right|
\]

\[
\left| B - A \right| = 0
\]

\[
\left| B \right| = \left| A \right| \text{ iff } B=A
\]

(b) Let \( A \) and \( B \) be disjoint sets, then:

\[
\left| A \cap B \right| = 0
\]

\[
\left| A \cup B \right| = \left| A \right| + \left| B \right|
\]

\[
\left| A - B \right| = \left| A \right|
\]

(C) \[ |\overline{A}| = |U - A| = |U| - |A| \]
Example: In a group of 42 tourists, everyone speaks English or French; there are 35 English speakers and 18 French speakers. How many speak both English & French?

\[ |U| = 42 = |E \cup F| \]

\[ E = \text{English speaker} \]
\[ F = \text{French speaker} \]

\[ |E \cup F| = |E| + |F| - |E \cap F| \]
\[ 42 = 35 + 18 - |E \cap F| \]
\[ \therefore |E \cap F| = 11 \]

HINT (General Strategy):
1) Define sets of things described in the prompt
2) Translate the question into a compound set
3) Compute cardinality of (2) with formulae

What if we have 3 sets: \( |A \cup B \cup C| \)?

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

L.H.S. = \[ |A \cup (B \cup C)| \] (association)
\[ = |A| + |B \cup C| - |A \cap (B \cup C)| \] (2-set cardinality)(distribution)
\[ = |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \]
\[ = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \]
\[ = R.H.S. \]

(can also be seen from the picture)
More than 3 sets?
Principle of Inclusion and Exclusion:

Given finite sets $A_1, A_2, \ldots, A_n$, $n \geq 2$, then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| =$$

$$+ \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Example: a survey of 150 college students reveals that $|U|=150$

- $|C|=83$ own Cars,
- $|B|=97$ own Bikes,
- $|H|=28$ own Motorcycles,
- $|C \cap B|=53$ own a car and a bike,
- $|C \cap H|=14$ own a car and a motorcycle,
- $|B \cap H|=7$ own a bike and a motorcycle,
- 2 own all three. \(= |C \cap B \cap H|\)
a. How many own a bike and nothing else?

\[ |B - (C \cup M)| \quad (|A-B| \text{ formula}) \]
\[ = |B| - |B \cap (C \cup M)| \quad (\text{distribution}) \]
\[ = |B| - |(B \cap C) \cup (B \cap M)| \quad (|A \cup B| \text{ formula}) \]
\[ = |B| - (|B \cap C| + |B \cap M| - |B \cap C \cap M|) \]
\[ = 97 - (53 + 7 - 2) \]
\[ = 39 \]

b. How many students do not own any of the three?

\[ 150 - |C \cup B \cup M| \]
\[ = 150 - (83 + 97 + 28 - 53 - 14 - 7 + 2) \]
\[ = 150 - 136 \]
\[ = 14 \]