Overview

• Study for HW#1 (Due next Tuesday in-class on 2/11)

• Last lecture
  – Proof templates for sets
  – Set operations: Union, Intersection, Difference, Sym-Diff
  – Cartesian Product

• Today’s lecture: Complete Lectures on Sets
  – More Proof Templates
    • Proof by cases,
    • Proof by using existing rules
    • Set identities: All *true* rules for sets that you can use for making your proof!
  – Basic counting theorem
  – Principle of inclusion and exclusion

Chapter 1.1 : Set Theory

Cond.
More proof templates (will show example later):

- **Proof by cases**: List all possible cases/situations. For each case, prove the conclusion separately.
- **Proof by using existing rules**: derive the desired conclusion from the assumption only by using the set identities.
- **Disproof by counterexample**: Find an $x$ that can be proved to demonstrate that the conclusion is false. (See example $A \neq B$)

Example: Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ by cases

One simple way: Use a membership table

1 means "$x$ is an element of this set"
0 means "$x$ is not an element of this set"

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$B \cap C$</th>
<th>$A \cup (B \cap C)$</th>
<th>$A \cup B$</th>
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<th>$(A \cup B) \cap (A \cup C)$</th>
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The membership values of LHS and RHS are same. Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
Proof by using existing rules

- **Deductive Proof**: Use the set identities (valid steps).
- Starting from a statement that you know is true.
- Revise the statement by using the identity rules successively, until you derive the form of statement that you want to prove.
- Each step provides a true statement if you only use proper identities! (One mistake makes the entire proof wrong!)
- HINT: think backward...

**Set Identities (properties)**

1. **Identity laws**
   \[ A \cup \emptyset = A, \ A \cap U = A \]

2. **Domination laws**
   \[ A \cup U = U, \ A \cap \emptyset = \emptyset \]

3. **Idempotent laws**
   \[ A \cup A = A, \ A \cap A = A \]

4. **Complementation law**
   \[ (A)' = A \]

5. **Commutative laws**
   \[ A \cup B = B \cup A, \ A \cap B = B \cap A \]

6. **Associative laws**
   \[ A \cup (B \cup C) = (A \cup B) \cup C, \ A \cap (B \cap C) = (A \cap B) \cap C \]

7. **More properties**
   \[ A \subseteq A \cup B, \ B \subseteq A \cup B, \ A \cap B \subseteq A, \ A \cap B \subseteq B \]
   \[ A \Rightarrow A \subseteq A \cup B, \ A \cap B \Rightarrow A \cap B \subseteq A \cup A \cap B \subseteq B \]
8. Distributive laws

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

9. De Morgan’s laws

\[ (A \cup B)^c = A^c \cap B^c \]
\[ (A \cap B)^c = A^c \cup B^c \]

Note: \( A^c = A' \)

10. Absorption laws

\[ A \cup (A \cap B) = A \]
\[ A \cap (A \cup B) = A \]

11. Complement laws

\[ A \cup A^c = U \]
\[ A \cap A^c = \emptyset \]

Note: You can prove all above identities. You may use basic definitions and existing identities to prove new set relations/properties

\[ A \cup \overline{A} = U \]
\[ A \cap \overline{A} = \emptyset \]
Use existing properties to prove the same theorem

(i) \( A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \) and
(ii) \( (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \)

(i) Suppose \( x \in A \cup (B \cap C) \). It is necessary to show that \( x \in (A \cup B) \cap (A \cup C) \) also

Since \( x \in A \cup (B \cap C) \), \( x \in A \) or \( x \in (B \cap C) \) (by \( \cup \) definition)

Note: There are 2 cases to prove

Case I, \( x \in A \), then,
\( x \in (A \cup B) \) and \( x \in (A \cup C) \) (by Identity 7 or text theorem 1(b))
It follows \( x \in (A \cup B) \cap (A \cup C) \) (by \( \cap \) definition)

Case II, \( x \in (B \cap C) \), then \( x \in B \) and \( x \in C \) (by \( \cap \) definition)
\( x \in (A \cup B) \) and \( x \in (A \cup C) \) (by Identity 7 or text theorem 1(b))
Therefore, \( x \in (A \cup B) \cap (A \cup C) \) (by \( \cap \) definition)

(ii) Suppose \( x \in (A \cup B) \cap (A \cup C) \). It is necessary to show that \( x \in A \cup (B \cap C) \).

Since \( x \in (A \cup B) \cap (A \cup C) \), then \( x \in (A \cup B) \) and \( x \in (A \cup C) \) (by \( \cap \) definition)
This means \( (x \in A \) or \( x \in B) \) and \( (x \in A \) or \( x \in C) \) (by \( \cup \) definition)
Note: There are 4 possible cases

Case I : if \( x \in A \) and \( x \in A \), then \( x \in (B \cap C) \)
(look at step 2 of Case I earlier)
Case II : if \( x \in A \) and \( x \in C \), then \( x \in A \cap C \)
(by \( \cap \) definition)
It follows \( x \in A \cup (B \cap C) \) (by Identity 7 or text theorem 2(b))
Case III, if \( x \in B \) and \( x \in A \),
Analogous to Case II
Case IV, if \( x \in B \) and \( x \in C \),
then \( x \in B \cap C \)
It follows \( x \in A \cap (B \cap C) \)
(by \( \cap \) definition)
(by Identity 7 or text theorem 1(b))
Basic Counting Theorems

A-B, \(A \cap B\), B-A are mutually exclusive/disjoint sets. i.e. \(x \in A-B\), then \(x \notin B\), and therefore \(x \notin B-A\), \(x \notin A \cap B\).

U: universal set

\[
\begin{align*}
|A \cup B| &= |A - B| + |B - A| + |A \cap B| \\
|A \cup B| &= |A| + |B| - |A \cap B| \\
|A - B| &= |A| - |A \cap B| \\
|B - A| &= |B| - |A \cap B|
\end{align*}
\]

\[|A \cap B|\] can be computed in several ways depends on the information given.

Other Counting Theorems

Let A and B be subsets of a finite universal set U

(a) Let B \(\subseteq\) A, then:

\[
|B| \leq |A|, \ |A-B| = |A| - |B| \\
|B| = |A| \text{ iff } B=A
\]

(b) Let A and B be disjoint sets, then:

\[
|A \cap B| = 0 \text{ and } |A \cup B| = |A| + |B|
\]

(C) \(|A'| = |U-A| = |U| - |A|\)
e.g. In a group of 42 tourists, everyone speaks English or French; there are 35 English speakers and 18 French speakers. How many speak both English & French?

\[ U = \text{group of tourists} \]
\[ E: \text{English Speakers} \]
\[ F: \text{French Speakers} \]

\[ |E \cup F| = 42 \]
\[ |E| = 35 \]
\[ |F| = 18 \]

\[ |E \cup F| = |E| + |F| - |E \cap F| \]
\[ 42 = 35 + 18 - |E \cap F| \]
\[ |E \cap F| = 11 \]

Therefore, \( |E \cap F| = 11 \)

For \( |A \cup B \cup C| \):

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

\[ \text{L.H.S.} = |A \cup (B \cup C)| \quad \text{(association)} \]
\[ = |A| + |B \cup C| - |A \cap (B \cup C)| \quad \text{(2-set cardinality)(distribution)} \]
\[ = |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \]
\[ = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \]
\[ = \text{R.H.S.} \]

(can also be seen from the picture)
**Principle of Inclusion and Exclusion**

Given finite sets $A_1, A_2, \ldots, A_n$, $n \geq 2$, then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| =$$

$$+ \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

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**Example:** A survey of 150 college students reveals that

- $|C| = 83$ own Cars,
- $|B| = 97$ own Bikes,
- $|M| = 28$ own Motorcycles,
- $|C \cap B| = 53$ own a car and a bike,
- $|C \cap M| = 14$ own a car and a motorcycle,
- $|B \cap M| = 7$ own a bike and a motorcycle,
- 2 own all three.

$$= |C \cap B \cap M|$$
a. How many own a bike and nothing else?

\[ |B - (C \cup M)| \ (|A-B| \text{ formula}) \]
\[ = |B| - |B \cap (C \cup M)| \ (\text{distribution}) \]
\[ = |B| - |(B \cap C) \cup (B \cap M)| \ (|A \cup B| \text{ formula}) \]
\[ = |B| - (|B \cap C| + |B \cap M| - |B \cap C \cap M|) \]
\[ = 97 - (53 + 7 - 2) \]
\[ = 39 \]

b. How many students do not own any of the three?

\[ 150 - |C \cup B \cup M| \]
\[ = 150 - (83 + 97 + 28 - 53 - 14 - 7 + 2) \]
\[ = 150 - 136 \]
\[ = 14 \]