Overview

- Study for HW#1 (Due this Thursday in-class 9/7)
- Last lecture
  - Proof templates for sets
  - Set operations: Union, Intersection, Difference, Sym-Diff
  - Cartesian Product
  - Set Identities: All *true* rules for sets that you can use for making your proof! (Continue Today)
- Today’s lecture
  - More Proof Templates
    - Proof by cases,
    - Proof by using existing rules/set identities
  - Basic counting theorem
  - Principle of inclusion and exclusion

Chapter 1.1 : Set Theory

Cond.
More proof templates (will show example later):

- **Proof by cases**: List all possible cases/situations. For each case, prove the conclusion separately.
- **Proof by using existing rules**: derive the desired conclusion from the assumption only by using the set identities.
- **Disproof by counterexample**: Find an x that can be proved to demonstrate that the conclusion is false. (See example A≠B)

Example: Prove \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \) by cases

One simple way: Use a membership table

1 means “x is an element of this set”
0 means “x is not an element of this set”

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<th>A</th>
<th>B</th>
<th>C</th>
<th>B ∩ C</th>
<th>A ∪ (B ∩ C)</th>
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<th>(A ∪ B) ∩ (A ∪ C)</th>
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Proof by using existing rules

- **Deductive Proof**: Use the set identities (valid steps).
- Starting from a statement that you know is true.
- Revise the statement by using the identity rules successively, until you derive the form of statement that you want to prove.
- Each step provides a true statement if you only use proper identities!
- HINT: think backward...

Set Identities (properties)

1. **Identity laws**
   \[ A \cup \emptyset = A, \quad A \cap U = A \]

2. **Domination laws**
   \[ A \cup U = U, \quad A \cap \emptyset = \emptyset \]

3. **Idempotent laws**
   \[ A \cup A = A, \quad A \cap A = A \]

4. **Complementation law**
   \[ (A')' = A \]

5. **Commutative laws**
   \[ A \cup B = B \cup A, \quad A \cap B = B \cap A \]

6. **Associative laws**
   \[ A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C \]

7. **More properties**
   \[ A \subseteq A \cup B, \quad B \subseteq A \cup B, \quad A \cap B \subseteq A, \quad A \cap B \subseteq B \]
   \[ A \Rightarrow A \subseteq A \cup B \]
   \[ A \cap B \Rightarrow A \cap B \subseteq A \cup \overline{A} \cap \overline{B} \subseteq B \]
8. Distributive laws

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

9. De Morgan’s laws

\[ (A \cup B)^c = A^c \cap B^c \]
\[ (A \cap B)^c = A^c \cup B^c \]

Note: \( A^c = A' \)

10. Absorption laws

\[ A \cup (A \cap B) = A \]
\[ A \cap (A \cup B) = A \]

11. Complement laws

\[ A \cup A^c = U \]
\[ A \cap A^c = \emptyset \]

Note: You can prove all above properties. You may use basic definitions and existing properties to prove new properties

\[ A \cup \overline{A} = U \]
\[ A \cap \overline{A} = \emptyset \]
Use existing properties to prove the same theorem:

(i) \( A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \)

(ii) \( (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \)

(i) Suppose \( x \in A \cup (B \cap C) \). It is necessary to show that \( x \in (A \cup B) \cap (A \cup C) \).

Since \( x \in A \cup (B \cap C) \), \( x \in A \) or \( x \in (B \cap C) \). (by definition)

Note: There are 2 cases to prove

Case I, \( x \in A \), then,
- \( x \in (A \cup B) \) and \( x \in (A \cup C) \) (by Identity 7 or text theorem 1(b))
- It follows \( x \in (A \cup B) \cap (A \cup C) \) (by \( \cap \) definition)

Case II, \( x \in (B \cap C) \), then \( x \in B \) and \( x \in C \) (by \( \cap \) definition)
- \( x \in (A \cup B) \) and \( x \in (A \cup C) \) (by Identity 7 or text theorem 1(b))
- Therefore, \( x \in (A \cup B) \cap (A \cup C) \) (by \( \cap \) definition)

(ii) Suppose \( x \in (A \cup B) \cap (A \cup C) \). It is necessary to show that \( x \in A \cup (B \cap C) \).

Since \( x \in (A \cup B) \cap (A \cup C) \), then \( x \in (A \cup B) \) and \( x \in (A \cup C) \) (by \( \cap \) definition)

This means \( x \in A \) or \( x \in B \) and \( x \in A \) or \( x \in C \) (by \( \cup \) definition)

Note: There are 4 possible cases

Case I: if \( x \in A \) and \( x \in A \) Look at step 2 of Case I earlier
then \( x \in A \)
then \( x \in A \cup (B \cap C) \) (by Identity 7 or text theorem 1(b))

Case II: if \( x \in A \) and \( x \in C \)
then \( x \in A \cap C \) (by \( \cap \) definition)
then \( x \in A \) (by Identity 7 or text theorem 1(b))
It follows \( x \in A \cup (B \cap C) \) (by Identity 7 or text theorem 1(b))

Case III, if \( x \in B \) and \( x \in A \),
Analogous to Case II

Case IV, if \( x \in B \) and \( x \in C \),
then \( x \in B \cap C \) (by \( \cap \) definition)
It follows \( x \in A \cup (B \cap C) \) (by Identity 7 or text theorem 1(b))
### Basic Counting Theorems

- A-B, A∩B, B-A are mutually exclusive/disjoint sets.
- i.e. \( x \in A-B \), then \( x \notin B \), and therefore \( x \notin B-A \) \( x \notin A \cap B \).

U: universal set

\[
\begin{align*}
|A \cup B| & = |A-B| + |B-A| + |A \cap B| \\
|A \cap B| & = |A| + |B| - |A \cap B| \\
|A-B| & = |A| - |A \cap B| \\
|B-A| & = |B| - |A \cap B|
\end{align*}
\]

∴ e.g. \( |A \cap B| \) can be computed in several ways depends on the information given.

### Other Counting Theorems

Let A and B be subsets of a finite universal set U

(a) Let \( B \subseteq A \), then:

\[
|B| \leq |A|, \quad |A-B| = |A| - |B|, \quad |B| = |A| \text{ iff } B=A
\]

(b) Let A and B be disjoint sets, then:

\[
|A \cap B| = 0 \quad \text{and} \quad |A \cup B| = |A| + |B|
\]

(C) \( |A'| = |U| - |A| \)
e.g. In a group of 42 tourists, everyone speaks English or French; there are 35 English speakers and 18 French speakers. How many speak both English & French?

\[|U| = 42, \quad |E| = 35, \quad |F| = 18\]

\[|E \cup F| = 42\]

\[|E \cap F| = ?\]

\[|E \cup F| = |E| + |F| - |E \cap F|\]

\[42 = 35 + 18 - |E \cap F|\]

\[|E \cap F| = 11\]

Therefore

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e.g. What if we have 3 sets: \(|A \cup B \cup C|\)?

\[|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|\]

L.H.S. = \(|A \cup (B \cup C)|\) (association)

\[= |A| + |B \cup C| - |A \cap (B \cup C)| \quad \text{(2-set cardinality)(distribution)}\]

\[= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)| - |A \cap B \cup (A \cap C)|\]

\[= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|\]

\[= \text{R.H.S.}\]

(can also be seen from the picture)
Principle of Inclusion and Exclusion

Given finite sets $A_1, A_2, \ldots, A_n$, $n \geq 2$, then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| =$$

$$+ \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Example: a survey of 150 college students reveals that

- $|U| = 150$ (all college students)
- $|C| = 83$ (own Cars)
- $|B| = 97$ (own Bikes)
- $|M| = 28$ (own Motorcycles)
- $|C \cap B| = 53$ (own a car and a bike)
- $|C \cap M| = 14$ (own a car and a motorcycle)
- $|B \cap M| = 7$ (own a bike and a motorcycle)
- $|C \cap B \cap M| = 2$ (own all three)
a. How many own a bike and nothing else?

\[ |B - (C \cup M)| \]  
\[ = |B| - |B \cap (C \cup M)| \] (distribution) 
\[ = |B| - |(B \cap C) \cup (B \cap M)| \] (|A \cup B| formula) 
\[ = |B| - (|B \cap C| + |B \cap M| - |B \cap C \cap M|) \] 
\[ = 97 - (53 + 7 - 2) \] 
\[ = 39 \]

b. How many students do not own any of the three?

\[ 150 - |C \cup B \cup M| \] 
\[ = 150 - (83 + 97 + 28 - 53 - 14 - 7 + 2) \] 
\[ = 150 - 136 \] 
\[ = 14 \]