Overview

• Pickup HW1 handout! HW1 due in one week.

• Last lecture
  – Set Definitions: Equality, Cardinality, Subset, Proper Subset
  – Tool: Venn diagram,
  – Set Definition: Power set

• Today’s lecture
  – Proof templates for sets: I and II
  – Set Operations: Union, Intersection, Difference, Sym Diff., Complement, Cartesian products
  – Set Identities

Chapter 1.1 : Set Theory
Cond.
Several Proof Templates for Sets

• \( x \in A \): show that \( x \) has the property that defines membership of \( A \).

Example: Let \( A = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \} \), show that \( x = 8 \in A \):

\[ x = 8 = 2k_0 \Rightarrow 4 = k_0 \]

Since \( 4 \in \mathbb{N} \), therefore \( 8 \in A \)

• \( A \subseteq B \): show that every element of \( A \) is also in \( B \). For \( A \subset B \): also show that some element \( x \) of \( B \) is not in \( A \).

Example: Let \( B = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \} \) and \( A = \{ n : n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \} \), show that \( A \subseteq B \):

Let \( x \) be an arbitrary element of \( A \), i.e. \( x = 4t_0 + 6 \in A \), (your goal is to show that this property also holds the property of \( B \) too!), first property of \( B \) is obvious.

Second property of \( B \):

\[ x = 4t_0 + 6 = 2(2t_0 + 3) = 2k_0 \text{ where } k_0 = 2t_0 + 3 \]

Since \( t_0 \in \mathbb{N} \), it follows that \( k_0 \in \mathbb{N} \), thus \( x = 2k_0 \in B \) is true, therefore \( A \subseteq B \)

• \( A = B \): show that \( A \subseteq B \) and \( B \subseteq A \).

Example: try yourself

• \( A \neq B \): show that \( A \notin B \) (i.e. some element \( x \) of \( A \) is not in \( B \)) or \( B \notin A \) (i.e. some element \( x \) of \( B \) is not in \( A \)).

Example: Let \( B = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \} \) and \( A = \{ n : n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \} \), show that \( B \not\subseteq A \):

Let \( x = 4 = 2*2 \in B \), (Must find an element by yourself)

Each element of \( A \) must satisfies \( 4t_0 + 6 \) for some \( t_0 \in \mathbb{N} \). Thus \( A = \{ 6, 10, 14, 18, ... \} \)

Clearly, \( x = 4 \notin A \)

Therefore, \( B \not\subseteq A \)

Therefore, \( A \neq B \)

• If \( A \), then \( B \): suppose \( A \) is true then derive \( B \) from the supposition

\[ A \Rightarrow B \]

• \( A \) if and only if \( B \): show “if \( A \), then \( B \)” and “if \( B \), then \( A \)”

Example: try yourself

\[ A \Rightarrow B \land B \Rightarrow A \]
Set Operations

1. Union
   \[ A \cup B = \{x \mid x \in A \text{ or } x \in B\} \]

   Example: \( A = \{a, b\}, B = \{b, c, d\} \)
   \( A \cup B = \{a, b, c, d\} \)

2. Intersection
   \[ A \cap B = \{x \mid x \in A \text{ and } x \in B\} \]

   Example: \( A = \{a, b\}, B = \{b, c, d\} \)
   \( A \cap B = \{b\} \)

   Note: Cardinality: \( |A \cup B| = |A| + |B| - |A \cap B| \)

   What if \( A \) and \( B \) are mutually exclusive/disjoint?

3. Difference
   \[ A - B = \{x \mid x \in A \text{ and } x \not\in B\} \]

   Example: \( A = \{a, b\}, B = \{b, c, d\} \)
   \( A-B = \{a\} \quad B-A? \)

4. Complement
   \[ A' = \{x \mid x \in U \text{ and } x \not\in A\} \]

   where \( U \) is the Universal Set

   Example: \( U = \mathbb{N}, B = \{250, 251, 252, \ldots\} \)
   \( B' = \{0, 1, 2, \ldots, 248, 249\} \)

   Note: Cardinality: \( |A-B| = |A| - |A \cap B|, \text{ M.E.?} \)
Venn Diagrams

Let U be universal set in all diagrams

- **Union** $A \cup B$
  \[ |A \cup B| = |A| + |B| - |A \cap B| \]

- **Intersection** $A \cap B$
  \[ |A \cap B| = 0 \]

- **Difference** $A - B$
  \[ |A - B| = |A| - |A \cap B| \]

- **Complement** $A'$

5. **Symmetric difference** $A \oplus B = (A - B) \cup (B - A)$

Cardinality?
\[
|A \oplus B| = |A| + |B| - 2|A \cap B| = |A \cup B| - |A \cap B| = |A| - |A \cap B| + |B - A|
\]

Note: When two sets are disjoint, $A \cap B = \emptyset$, we have
\[ |A \oplus B| = |A \cup B| = |A| + |B| \]
Example Problem: On Compound Sets

A = {1, 2, 3, 5, 10}, B = {2, 4, 7, 8, 9}
C = {5, 8, 10}, U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

A ∪ B = {1, 2, 3, 4, 5, 7, 8, 9, 10}; A − C = {1, 2, 3}
B′ ∩ (A ∪ C) = {1, 3, 5, 6, 10} ∩ {1, 2, 3, 5, 8, 10} = {1, 3, 5, 10};
B ⊕ C = {2, 4, 7, 9} ∪ {5, 10} = {2, 4, 5, 7, 9, 10};

What is the compound set of ...?

6. Cartesian Product

A × B = {(x, y) | x ∈ A and y ∈ B}

Example: A = {a, b}, B = {x, y, z}
A × B = {(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)}

Note that:
• For non-empty sets A and B: A ≠ B ⇔ A × B ≠ B × A
• A × ∅ = ∅
• ∅ × A = ∅
• |A × B| = |A| × |B|

• The Cartesian product of two or more sets is defined as:
A₁ × A₂ × ... × Aₙ = {(a₁, a₂, ..., aₙ) | aᵢ ∈ Aᵢ for 1 ≤ i ≤ n}
Cardinality?
|A₁ × A₂ × ... × Aₙ| = |A₁| × |A₂| × ... × |Aₙ|
Set Identities (properties)

1. **Identity laws**
   \[ A \cup \emptyset = A, \ A \cap U = A \]

2. **Domination laws**
   \[ A \cup U = U, \ A \cap \emptyset = \emptyset \]

3. **Idempotent laws**
   \[ A \cup A = A, \ A \cap A = A \]

4. **Complementation law**
   \[ (A')' = A \]

5. **Commutative laws**
   \[ A \cup B = B \cup A, \ A \cap B = B \cap A \]

6. **Associative laws**
   \[ A \cup (B \cup C) = (A \cup B) \cup C, \ A \cap (B \cap C) = (A \cap B) \cap C \]

7. **More properties**: 
   \[ A \subseteq A \cup B, \ B \subseteq A \cup B, \ A \cap B \subseteq A, \ A \cap B \subseteq B \]
   \[ A \Rightarrow A \subseteq A \cup B \]
   \[ A \cap B \Rightarrow A \cap B \subseteq A \cup B \]

8. **Distributive laws**
   \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
   \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

9. **De Morgan’s laws**
   \[ (A \cup B)^c = A^c \cap B^c \]
   \[ (A \cap B)^c = A^c \cup B^c \]
   \[ \text{Note: } A^c = A' \]
10. Absorption laws

\[ A \cup (A \cap B) = A \]
\[ A \cap (A \cup B) = A \]

11. Complement laws

\[ A \cup A^c = U \]
\[ A \cap A^c = \emptyset \]

Note: You can prove all above identities. You may use basic definitions and existing identities to prove new set relations/properties.

\[ \overline{A \cup A} = \emptyset \]
\[ \overline{A \cap A} = U \]

Several Proof Templates

• \( x \in A \): show that \( x \) has the property that defines membership of \( A \).

  Example: Let \( A = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \} \), show that \( x=8 \in A \):
  
  \[ x=8 \text{ is obviously } x \in \mathbb{N} \text{ (so the first property of } A \text{ holds)} \]
  
  \[ 8 = 2k_0 \Rightarrow 4 = k_0 \]
  
  Since \( 4 \in \mathbb{N} \) (the second property of \( A \) holds), (thus) \( x = 8 \in A \)

• \( A \subseteq B \): show that every element of \( A \) is also in \( B \).

  Example: Let \( B = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \} \) and
  \( A = \{ n : n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \} \), show that \( A \subseteq B \):
  
  Let \( x \) be an arbitrary element of \( A \), i.e. \( x = 4t + 6, x \in \mathbb{N} \)
  
  (your goal is to show that \( x \) also holds the property of \( B \) too!)
  
  first property is obvious. For second, \( x = 4t + 6 = 2(2t + 3) = 2k_0 \) where \( k_0 = 2t + 3 \)
  
  clearly, \( k_0 \in \mathbb{N} \) thus \( x = 4t + 6 = 2k_0 \in B \), therefore \( A \subseteq B \)

• \( A \subset B \): show \( A \subseteq B \) and also show that some element \( x \) of \( B \) is not in \( A \)
• **A = B**: show that $A \subseteq B$ and $B \subseteq A$

Example: try yourself

• **A ≠ B**: show that $A \subseteq B$ (i.e. some element $x$ of $A$ is not in $B$) or $B \subseteq A$ (i.e. some element $x$ of $B$ is not in $A$)

Example: Let $B = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \}$ and $A = \{ n : n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \}$, show that $B \not\subseteq A$:

  - Let $x = 4 = 2 \times 2 \in B$, (Must find an element by yourself)
  - Each element of $A$ must satisfies $4t_o + 6$ for some $t_o \in \mathbb{N}$. Thus $A = \{ 6, 10, 14, 18, \ldots \}$
  - Clearly, $x = 4 \notin A$
  - Therefore, $B \not\subseteq A$

• **If A, then B**: suppose $A$ is true then derive $B$ from the supposition
• **A if and only if B**: show “if A, then B” and “if B, then A”

Example: try yourself

$A \iff B$ show $(A \Rightarrow B) \land (B \Rightarrow A)$