Overview

• Last Lecture on Set Theory
  – Definition: an unordered collection of distinct discrete objects
  – Notation:
    • Memberships: $a \in A, a \notin A$
    • Enumeration: e.g., \{A, 5, \{cat, dog\}, car, o\}
    • Property description: e.g., \{x | x is odd & x is prime\}

• Today’s Lecture on Set Theory
  – Standard sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \emptyset$
  – Equality: $A = B$ if and only if they contain exactly same elements
  – Subset
  – Cardinality (Size of Sets)
  – Venn Diagram
  – Power set
  – Proof Templates

Chapter 1.1 : Set Theory
Cond.
Standard sets (or Special Sets):

- \( \mathbb{N} \) = set of natural numbers or set of all nonnegative integers 
  \( \{0,1,2,3,\ldots\} \)
- \( \mathbb{Z} \) = set of all integers \( \{-\ldots,-2,-1,0,1,2,3,\ldots\} \)
- \( \mathbb{Q} \) = set of all rational numbers or the set of fractions of integer with nonzero denominator, i.e. \( \{m/n \mid m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0\} \).
  E.g. \( 5/22 \), \(-2568/535 \)
- \( \mathbb{R} \) = set of all real numbers. E.g. \( 9.5656282, -5.6723, \) or \( 0.333333 \ldots \)
- \( \emptyset \) = the empty set or the set \{ \} with no elements

- Use special sets to describe elements in a set
  e.g. \( S = \{x \mid x \in \mathbb{N} \text{ and } x < 0\} \), then \( S = \emptyset = \{\} \)
  e.g. \( A = \{x \mid x \in \mathbb{N} \text{ and } x > 10 \text{ and } x \leq 20\} = \{11,12,13,\ldots,20\} \)

Set Equality:

Sets \( A \) and \( B \) are equal if and only if they contain exactly the same elements.

Examples:

- \( A = \{9, 2, 7, -3\}, B = \{7, 9, -3, 2\} : A = B \)
- \( A = \{\text{dog, cat, horse}\}, B = \{\text{cat, horse, rabit, dog}\} : A \neq B \)
- \( A = \{x \mid x \in \mathbb{Z} \text{ and } x^2 - x - 2 = 0\} , B = \{2, -1\} : A = B \)
  \[
  (x+1)(x-2) = 0 \\
  x = 2, -1
  \]
Subsets:

Given $A = \{a_1, a_2, \ldots, a_n\}$ if $a_1 \in B, a_2 \in B, \ldots, a_n \in B$ then $A \subseteq B$

$A$ is a subset of $B$, denoted by $A \subseteq B$, if every element of $A$ is also an element of $B$.

$A$ is a proper subset of $B$, denoted by $A \subset B$, if $A \subseteq B$ and $A \neq B$.

$\emptyset \subseteq A$ for any set $A$ (see text theorem 1)

$A = B \rightarrow A \subseteq B$

$A \neq B$ for any set $A$ (but $\emptyset \in A$ may not hold for any set $A$), Q: when $\emptyset \in A$ true?

$A \subseteq A$ for any set $A$ (see text theorem 1)

Examples: $A = \{1, 7, 9, 15\}$ and $B = \{7, 9\}$ then list all possible subset relations.

$\emptyset \subseteq A$

$\emptyset \subseteq A, \emptyset \subseteq B$

Rules on Subsets:

\begin{itemize}
  \item $A = B \iff (A \subseteq B) \land (B \subseteq A)$
  \item $(A \subseteq B) \land (B \subseteq C) \Rightarrow A \subseteq C$
\end{itemize}

Note: $\land$ “and”

$\iff$ “if and only if” or “iff”

$\Rightarrow$ “if... then...”

$a \leq b \iff a \leq b$ and $b \leq q$

If $a \leq b$ and $b \leq c$ then $a \leq c$

Similar pattern!

But not the same.

Just use these “translation” for now.

logical connections to be explained later...
Size of Sets:

{A, {cat, dog}, ∅, 1035, {1,2,3,...}}?
{1,2,3,4,5}? 

How many elements in a set? (intuitive definition)

Finite set: a set with no elements or its elements can be matched with the elements of some subset \{0,1,2,...,n\} of \(\mathbb{N}\). Informally, a set whose count stop somewhere.

Infinite set: is a set which is not finite.

Cardinality of Sets:

If a set \(S\) contains \(n\) distinct elements \((n \in \mathbb{N})\), Cardinality of the set \(S\), denoted by \(|S|\), is \(n\). \(|S| = n\)

Examples:

\(A = \{Mercedes, BMW, Porsche\}, \quad |A| = 3\)
\(B = \{1, \{2, 3\}, \{4, 5\}, 6\}, \quad |B| = ?\)
\(C = \emptyset, \quad |C| = ?\)
\(D = \{x : x \in \mathbb{N} \text{ and } x \leq 7000\}, \quad |D| = ?\)
\(E = \{x \mid x \in \mathbb{N} \text{ and } x \geq 7000\}, \quad |E| = ?\)
\(F = \{\emptyset\}, \quad |F| = ?\)
\(G = \{1,1,1,1,1\}, \quad |G| = ?\)
\(H = \{0,1,2,3,...\}, \quad |H| = ?\)
\(I = \{0,1,2,3,...\}, \quad |I| = ?\)
Venn Diagram:

• To illustrate set-theoretic relationship in a diagram
• $U =$ universal set (set of all elements) that you define (can be $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{R}$ or anything!), denoted by a box

Given a set $A$

1. $A \subseteq U$ (a circle in side the U-box)
2. $A = U$

Given two sets, $A$ & $B$

1. $A = \{1,2\}, B = \{3,4\}$
2. $A = \{1,2,3\}, B = \{3,4\}$
3. $A \subseteq B, (A = \{1,2\}, B = \{1,2,3,4\})$
4. $A = B, (A = \{1,2\}, B = \{1,2\})$

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Venn Diagram:

• Example:
$U =$ universal set (set of all elements)
$A \subseteq B$ and $B \subseteq C$
Power sets:

Power set of $S$, denoted by $P(S)$ or $2^S$, contains all possible subsets of $S$.

$\therefore \emptyset \in P(S)$ and $S \in P(S)$ are always true.

Example: $S = \{1, 2, 3\}$

$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, $|P(S)| = 8$

Example: $T = \{\text{dog, cat}\}$

$P(T) = \{\emptyset, \{\text{dog}\}, \{\text{cat}\}, \{\text{dog, cat}\}\}$, $|P(T)| = 4$

Example: $A = \emptyset$

$P(A) = \{\emptyset\}$

$= \{\emptyset, A\} = \{\emptyset, \emptyset\} = \{\emptyset\}$

Note: $|A| = 0$, $|P(A)| = 1$

• Cardinality of power sets: $|P(A)| = 2^{|A|}$

Imagine each element in $A$ has an “on/off” switch. Each possible switch configuration in $A$ corresponds to one subset of $A$, thus one element in $P(A)$, e.g., $A = \{x, y, z\}$

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$8 = 2^3 = \frac{2^3}{2^0}$

$32 = 2^5 = \frac{2^5}{2^0}$
Proof Templates for Sets

• What is proof?
  – **Proof**: Use mathematically valid arguments to conclude that a given statement is True.
  – **Disproof**: Use mathematically valid arguments to conclude that a given statement is *False*.

• Templates
  – \( x \in A \)
  – \( A \subseteq B \)
  – \( A \subset B \)
  – \( A = B \)
  – \( A \neq B \)
  – If \( A \), then \( B \), \( A \) if and only if \( B \)

Several Set Proof Templates

• \( x \in A \): show that \( x \) has the property that defines membership of \( A \).
  
  Example: Let \( A = \{ n \mid n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \} \), show that \( x = 8 \in A \):
  
  \( P_1 \): \( x = 8 \) is obviously \( x \in \mathbb{N} \) (so the first property of \( A \) holds)
  
  \( P_2 \): \( x = 8 = 2k_0 \Rightarrow 4 = k_0 \)

  Since \( 4 \in \mathbb{N} \) (the second property holds), therefore \( 8 \in A \)

• \( A \subseteq B \): show that every element of \( A \) is also in \( B \).
  
  Example: Let \( B = \{ n \mid n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \} \) and \( A = \{ n \mid n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \} \), show that \( A \subseteq B \): Let \( x \) be an arbitrary element of \( A \), i.e. \( x = 4t_0 + 6 \in \mathbb{N} \), \( t_0 \in \mathbb{N} \) (your goal is to show that this property also holds the property of \( B \) too!), first property \( PB_1 \) of \( B \) is obvious.

  second property \( PB_2 \) of \( B \):
  
  \( x = 4t_0 + 6 = 2(2t_0 + 3) = 2k_0 \) where \( k_0 = 2t_0 + 3 \)
  
  since \( t_0 \in \mathbb{N} \) it follows that \( k_0 \in \mathbb{N} \),
  
  thus \( x = 2k_0 \in B \) is true, therefore \( A \subseteq B \)

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• $A \subseteq B$ : show $A \subseteq B$ and some element $x$ of $B$ is not in $A$
• $A = B$ : show that $A \subseteq B$ and $B \subseteq A$

Example : try yourself

• $A \neq B$ : show that $A \not\subseteq B$ (i.e. some element $x$ of $A$ is not in $B$)
or $B \not\subseteq A$ (i.e. some element $x$ of $B$ is not in $A$)

Example: Let $B = \{ n \mid n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \}$ and $A = \{ n \mid n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \}$,
show that $B \not\subseteq A$ :

Let $x = 4 = 2^2 \in B$, (Must find an element by yourself)
Each element of $A$ must satisfies $4t_0 + 6$ for
some $t_0 \in \mathbb{N}$ . Thus $A = \{6, 10, 14, 18, ... \}$
clearly, $x = 4 \not\in A$
Therefore, $B \not\subseteq A$
Therefore, $A \neq B$ □

• If $A$, then $B$ : suppose $A$ is true then derive $B$ from the supposition
• $A$ if and only if $B$ : show “if $A$, then $B$” and “if $B$, then $A$”

Example : try yourself

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