Overview

• Last Lecture on Set Theory
  – Definition: an unordered collection of distinct discrete objects
  – Notation:
    • Memberships: \( a \in A, a \notin A \)
    • Enumeration: e.g., \( \{A,5,\{cat, dog\}, car, o\} \)
    • Property description: e.g., \( \{x| x \text{ is odd } \& \text{ x is prime} \} \)

• Today’s Lecture on Set Theory
  – Standard sets: \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \emptyset \)
  – Equality: \( A=B \) if and only if they contain exactly same elements
  – Subset
  – Cardinality (Size of Sets)
  – Venn Diagram
  – Power set
  – Proof Templates

Chapter 1.1 : Set Theory
Cond.
Standard sets (or Special Sets):

\( \mathbb{N} \) = set of natural numbers or set of all nonnegative integers 
\( \{0,1,2,3,\ldots\} \)

\( \mathbb{Z} \) = set of all integers \( \{\ldots,-3,-2,-1,0,1,2,3,\ldots\} \)

\( \mathbb{Q} \) = set of all rational numbers or the set of fractions of integer with nonzero denominator, i.e. \( \{m/n : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0\} \).

E.g. \( 5/22 \), \( -2568/535 \)

\( \mathbb{R} \) = set of all real numbers. E.g. \( 9.5656282 \), \( -5.6723 \), or \( 0.333333 \ldots \)

\( \emptyset \) = the empty set or the set \( \{\} \) with no elements

• Use special sets to describe elements in a set
  e.g. \( S = \{x : x \in \mathbb{N} \text{ and } x < 0\} \), then \( S = \{\} = \emptyset \)
  e.g. \( A = \{x : x \in \mathbb{N} \text{ and } x > 10 \text{ and } x \leq 20\} = \{11,12,13,\ldots,20\} \)

Set Equality:

Sets \( A \) and \( B \) are equal if and only if they contain exactly the same elements.

Examples:
\( A = \{9, 2, 7, -3\} \), \( B = \{7, 9, -3, 2\} : A = B \)

\( A = \{\text{dog, cat, horse}\} \), \( B = \{\text{cat, horse, rabbit, dog}\} : A \neq B \)

\( A = \{x : x \in \mathbb{Z} \text{ and } x^2 - x - 2 = 0\} \), \( B = \{2, -1\} : A = B \)

\[ \begin{align*}
(x+1)(x-2) &= 0 \\
\therefore x &= 2, -1
\end{align*} \]
**Subsets:**

Given \( A = \{a_1, a_2, \ldots, a_n\} \), if \( a_1 \in B, a_2 \in B, \ldots, a_n \in B \) then \( A \subseteq B \).

A is a subset of \( B \), denoted by \( A \subseteq B \), if every element of \( A \) is also an element of \( B \).

A is a proper subset of \( B \), denoted by \( A \subset B \), if \( A \subseteq B \) and \( A \neq B \).

\( \emptyset \subseteq A \) for any set \( A \) (see text theorem 1)

(though \( \emptyset \in A \) may not hold for any set \( A \), Q: when \( \emptyset \in A \) true?)

\( A \subseteq A \) for any set \( A \) (see text theorem 1)

Examples: \( A = \{1, 7, 9, 15\} \) and \( B = \{7, 9\} \) then list all possible subset relations.

\[ B \subseteq A \\
B \subseteq A \\
A \subseteq A, B \subseteq B \\
\emptyset \subseteq A, \emptyset \subseteq B \\
\emptyset \subseteq A, \emptyset \subseteq B \]

**Rules on Subsets:**

- \( A = B \iff (A \subseteq B) \land (B \subseteq A) \)
- \( (A \subseteq B) \land (B \subseteq C) \Rightarrow A \subseteq C \)

Note: \( \land \) “and”

\( \iff \) “if and only if” or “iff”

\( \Rightarrow \) “if... then...”
**Size of Sets:**

{A, {cat, dog}, ∅, 1035, {1,2,3,...}}?

{1,2,3,4,5}?

How many elements in a set? (intuitive definition)

Finite set: a set with no elements or its elements can be matched with the elements of some subset \{0,1,2,...,n\} of \mathbb{N}. Informally, a set whose count stop somewhere

Infinite set:

is a set which is not finite

**Cardinality of Sets:**

If a set \( S \) contains \( n \) distinct elements \( (n \in \mathbb{N}) \), Cardinality of the set \( S \), denoted by \( |S| \), is \( n \). \(|S| = n\)

Examples:

\( A = \{Mercedes, BMW, Porsche\}, \ |A| = 3 \)

\( B = \{1, \{2, 3\}, \{4, 5\}, 6\}, \ |B| = ? \)

\( C = \emptyset, \ |C| = ? \)

\( D = \{x : x \in \mathbb{N} \text{ and } x \leq 7000 \}, \ |D| = ? \)

\( E = \{x \mid x \in \mathbb{N} \text{ and } x \geq 7000 \}, \ |E| = ? \)

\( F = \emptyset\), \(|F| = ? \)

\( G = \{1, 1, 1, 1\}, \ |G| = ? \)

\( H = \{0,1,2,3,...\}, \ |H| = ? \)

\( I = \{0,1,2,3,...\}, \ |I| = ? \)
Venn Diagram:

• To illustrate set-theoretic relationship in a diagram
• \( U \) = universal set (set of all elements) that you define (can be \( \mathbb{N} \) or \( \mathbb{Z} \) or \( \mathbb{R} \) or anything!), denoted by a box

Given a set \( A \)
1. \( A \subseteq U \) (a circle inside the U-box)
2. \( A = U \)

Given two sets, \( A \) & \( B \)
1. \( A = \{1, 2\}, B = \{3, 4\} \)
2. \( A = \{1, 2, 3\}, B = \{3, 4\} \)
3. \( A \subseteq B, (A = \{1, 2\}, B = \{1, 2, 3, 4\}) \)
4. \( A = B, (A = \{1, 2\}, B = \{1, 2\}) \)

Venn Diagram:

• Example:
\( U \) = universal set (set of all elements)
\( A \subseteq B \) and \( B \subseteq C \)
**Power sets:**

**Power set of S**, denoted by \( P(S) \) or \( 2^S \), contains all possible subsets of \( S \).

\[ \varnothing \in P(S) \text{ and } S \in P(S) \text{ are always true.} \]

Example: \( S = \{1, 2, 3\} \)

\[ P(S) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} \]

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Example: \( T = \{\text{dog, cat}\} \)

\[ P(T) = \{\varnothing, \{\text{dog}\}, \{\text{cat}\}, \{\text{dog, cat}\}\} \]

Example: \( A = \varnothing \)

\[ P(A) = \{\varnothing\} \]

\[ = \{\varnothing, A\} = \{\varnothing, \varnothing\} = \{\varnothing\} = \{\} \]

Note: \( |A| = 0 \), \( |P(A)| = 1 \)

**Cardinality of power sets:** \( |P(A)| = 2^{|A|} \)

Imagine each element in \( A \) has an “on/off” switch. Each possible switch configuration in \( A \) corresponds to one subset of \( A \), thus one element in \( P(A) \), e.g., \( A = \{x, y, z\} \)

\[ 2^3 = 8 \]

\[ |P(\{x, y, z\})| = 2^3 \]

Create all possible on/off combinations & count them!
Proof Templates for Sets

• What is proof?
  – **Proof**: Use mathematically valid arguments to conclude that a given statement is True.
  – **Disproof**: Use mathematically valid arguments to conclude that a given statement is False.

• Templates
  – $x \in A$
  – $A \subseteq B$
  – $A \subset B$
  – $A = B$
  – $A \neq B$
  – If $A$, then $B$, $A$ if and only if $B$

Several Set Proof Templates

• $x \in A$ : show that $x$ has the property that defines membership of $A$.
  
  Example: Let $A = \{ n \mid n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \}$, show that $x = 8 \in A$:
  
  $P_x$: $x = 8$ is obviously $x \in \mathbb{N}$ (so the first property of $A$ holds)
  $P_2$: $x = 2k_o \Rightarrow 4 = k_o$
  
  Since $4 \in \mathbb{N}$ (the second property holds), therefore $8 \in A$.

• $A \subseteq B$ : show that every element of $A$ is also in $B$.
  
  Example: Let $B = \{ n \mid n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \}$ and $A = \{ n \mid n \in \mathbb{N} \text{ and } n = 4t + 6 \text{ for some } t \in \mathbb{N} \}$, show that $A \subseteq B$:
  
  Let $x$ be an arbitrary element of $A$, i.e. $x = 4t_o + 6 \in \mathbb{N}$, $t_o \in \mathbb{N}$ (your goal is to show that this property also holds the property of $B$ too!), first property $PB_1$ of $B$ is obvious.
  
  Second property $PB_2$ of $B$:
  $x = 4t_o + 6 = 2(2t_o + 3) = 2k_o$ where $k_o = 2t_o + 3$
  
  Since $t_o \in \mathbb{N}$ it follows that $k_o \in \mathbb{N}$,
  
  thus $x = 2k_o \in B$ is true, therefore $A \subseteq B$.
• $A \subseteq B$ : show $A \subseteq B$ and some element $x$ of $B$ is not in $A$

• $A = B$ : show that $A \subseteq B$ and $B \subseteq A$

Example : try yourself

• $A \neq B$ : show that $A \nsubseteq B$ (i.e. some element $x$ of $A$ is not in $B$)
or $B \nsubseteq A$ (i.e. some element $x$ of $B$ is not in $A$)

Example: Let $B = \{ n | n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \}$ and $A = \{ n | n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \}$,

show that $B \nsubseteq A$ :

Let $x = 4 = 2*2 \in B$, (Must find an element by yourself)

Each element of $A$ must satisfies $4t_0 + 6$ for some $t_0 \in \mathbb{N}$ . Thus $A = \{ 6, 10, 18, \ldots \}$

clearly, $x = 4 \notin A$

Therefore, $B \nsubseteq A$

Therefore, $A \neq B$ □

• If $A$, then $B$ : suppose $A$ is true then derive $B$ from the supposition

• $A$ if and only if $B$ : show “if $A$, then $B$" and “if $B$, then $A$”

Example : try yourself