Overview

• Last Lecture on Set Theory
  – Definition: an unordered collection of discrete objects
  – Notation: enumeration & property description, membership
  – Standard sets: N, Z, Q, R, ∅
  – Equality: A=B if and only if they contain exactly same elements

• Today’s Lecture on Set Theory
  – Subset
  – Cardinality (Size of Sets)
  – Venn Diagram
  – Power set
  – Proof Templates

Chapter 1.1 : Set Theory
Cond.
A is a **subset** of B, denoted by $A \subseteq B$, if every element of A is also an element of B.

A is a **proper subset** of B, denoted by $A \subset B$, if $A \subseteq B$ and $A \neq B$.

$\emptyset \subseteq A$ for any set A (see text theorem 1) (but $\emptyset \in A$ may not hold for any set A), when $\emptyset \subseteq A$ true?

$A \subseteq A$ for any set A (see text theorem 1)

Examples: A = {1, 7, 9, 15} and B = {7, 9} then list all possible subset relations.

<table>
<thead>
<tr>
<th>$\subseteq$</th>
<th>$\subset$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \subseteq B$</td>
<td>$B \subseteq A$</td>
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<td>$\emptyset \subseteq A$</td>
<td>$\emptyset \subseteq B$</td>
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**Size of Sets**

{1, {1, 2, 3}, cat, \$dog, \{elephant\}, 1, 2, 3}  

How many elements in a set? (intuitive definition)

**Finite set**: a set with no elements or its elements can be matched with the elements of some subset \{0,1,2,...,n\} of N. Informally, a set whose count stop somewhere

**Infinite set**: is a set which is not finite
Cardinality of Sets

If a set S contains \( n \) “distinct” elements, \( n \in \mathbb{N} \), Cardinality of the set S, denoted by \( |S| \), is \( n \).

Examples:
A = \{ Mercedes, BMW, Porsche \}, \( |A| = 3 \)
B = \{ 1, \{ 2, 3 \}, \{ 4, 5 \}, 6 \}, \( |B| = ? \)
C = \emptyset, \( |C| = ? \)
D = \{ \text{x : x} \in \mathbb{N} \text{ and } x \leq 7000 \}, \( |D| = ? \)
E = \{ \text{x : x} \in \mathbb{N} \text{ and } x \geq 7000 \}, \( |E| = ? \)
F = \{ \emptyset \}, \( |F| = ? \)
G = \{1,1,1,1,1\}, \( |G| = ? \)
H = \{0, 1, 2, 3, ...\}, \( |H| = ? \)
I = \{\{0,1,2,3,...\}\}, \( |I| = ? \)

Simple rules/definitions:

• \( A = B \iff (A \subseteq B) \land (B \subseteq A) \)
• \( (A \subseteq B) \land (B \subseteq C) \implies A \subseteq C \)

Note: \( \land \) “and”
\( \iff \) “if and only if” or “iff”
\( \implies \) “if... then...”
Venn Diagram

- To illustrate set-theoretic relationship in a diagram
- \( U = \) universal set (set of all elements) that you define (can be \( \mathbb{N} \) or \( \mathbb{Z} \) or \( \mathbb{R} \) or anything!), denoted by a box

Given a set \( A \)
1. \( A \subseteq U \) (a circle in side the U-box)
2. \( A = U \)

Given two sets, \( A \) & \( B \)
1. \( A = \{1, 2\}, B = \{3, 4\} \)
2. \( A = \{1, 2, 3\}, B = \{3, 4\} \)
3. \( A \subseteq B, (A = \{1, 2\}, B = \{1, 2, 3, 4\}) \)
4. \( A = B, (A = \{1, 2\}, B = \{1, 2\}) \)

\[ \begin{align*}
1 & \quad \text{(A \subseteq U)} \\
2 & \quad \text{(A = U)} \\
3 & \quad \text{(A \subseteq B)} \\
4 & \quad \text{(A = B)} \\
\end{align*} \]

Venn Diagram

- Example:
  \( U = \) universal set (set of all elements)
  \( A \subseteq B \) and \( B \subseteq C \)
Power set of $S$, denoted by $P(S)$ or $2^S$, contains all of the subsets of $S$.

$\emptyset \in P(S)$ and $S \in P(S)$ are always true.

Example: $S = \{1, 2, 3\}$

$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

$|P(S)| = 8$

Example: $A = \emptyset$

$P(A) = \{\emptyset\}$. Note: $|A| = 0$, $|P(A)| = 1$

Cardinality of power sets: $|P(A)| = 2^{|A|}$

Imagine each element in $A$ has an “on/off” switch. Each possible switch configuration in $A$ corresponds to one subset of $A$, thus one element in $P(A)$, e.g., $A = \{x, y, z\}$

<table>
<thead>
<tr>
<th>$A$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>$z$</td>
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</tbody>
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What happens with $A = \{x, y, z, a, b\}$?
Several Proof Templates

• $x \in A$: show that $x$ has the property that defines membership of $A$.
  
  Example: Let $A = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \}$, show that $x=8 \in A$:
  
  $x=8$ is obviously $x \in \mathbb{N}$ (so the first property of $A$ holds)
  
  $x = 8 = 2k_0$ \Rightarrow $4 = k_0$
  
  Since $4 \in \mathbb{N}$ (the second property holds), therefore $8 \in A$

• $A \subseteq B$: show that every element of $A$ is also in $B$. For $A \subset B$: also show that some element $x$ of $B$ is not in $A$

  Example: Let $B = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \}$ and $A = \{ n : n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \}$, show that $A \subseteq B$:
  
  Let $x$ be an arbitrary element of $A$, i.e. $x = 4t_0+6 \in A$
  
  (your goal is to show that this property also holds the property of $B$ too!), first property of $B$ is obvious.
  
  second property of $B$: $x=4t_0+6 = 2(2t_0+3) = 2k_0$ where $k_0 = 2t_0+3$
  
  since $t_0 \in \mathbb{N}$ it follows that $k_0 \in \mathbb{N}$,
  
  thus $x = 2k_0 \in B$ is true, therefore $A \subseteq B$

• $A = B$: show that $A \subseteq B$ and $B \subseteq A$

  Example: try yourself

• $A \neq B$: show that $A \notin B$ (i.e. some element $x$ of $A$ is not in $B$) or $B \notin A$ (i.e. some element $x$ of $B$ is not in $A$)

  Example: Let $B = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \}$ and $A = \{ n : n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \}$, show that $B \notin A$:

  Let $x = 4 = 2*2 \in B$, (Must find an element by yourself)
  
  Each element of $A$ must satisfies $4t_0 + 6$ for some $t_0 \in \mathbb{N}$. Thus $A = \{ 6, 10, 14, ... \}$
  
  clearly, $x = 4 \notin A$
  
  Therefore, $B \notin A$
  
  Therefore, $A \neq B$

• If $A$, then $B$: suppose $A$ is true then derive $B$ from the supposition
• A if and only if B: show “if $A$, then $B$” and “if $B$, then $A$”

Example: try yourself