Overview

• Last Lecture on Set Theory
  – Definition: an unordered collection of distinct discrete objects
  – Notation: enumeration & property description, membership
  
  \{\text{A, 0, } x, \text{a, } \ldots \}
  \text{variable | property | and property \ldots }

• Today’s Lecture on Set Theory
  – Standard sets: \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \emptyset \)
  – Equality: \( A = B \) if and only if they contain exactly same elements
  – Cardinality (Size of Sets)
  – Subset
  – Venn Diagram
  – Power set
  – Proof Templates

Chapter 1.1 : Set Theory
Cond.
Standard sets (or Special Sets):

- \( \mathbb{N} \) = set of natural numbers or set of all nonnegative integers
  \( \{0, 1, 2, 3, \ldots\} \)
  - \( \infty \)
  - "ininitely many elements in that!"

- \( \mathbb{Z} \) = set of all integers \( \{-\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \)

- \( \mathbb{Q} \) = set of all rational numbers or the set of fractions of integers with nonzero denominator, i.e. \( \{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0 \} \).
  E.g. 5/22, -2568/535

- \( \mathbb{R} \) = set of all real numbers. E.g. 9.5656282, -5.6723, or 0.333333....

- \( \emptyset \) = the empty set or the set \( \{\} \) with no elements

- Use special sets to describe elements in a set
  e.g. \( S = \{ x : x \in \mathbb{N} \text{ and } x < 0 \} \), then \( S = \{\} = \emptyset \)
  e.g. \( A = \{ x \mid x \in \mathbb{N} \text{ and } x > 10 \text{ and } x \leq 20 \} = \{11, 12, 13, \ldots, 20\} \)

Set Equality:

Sets \( A \) and \( B \) are equal if and only if they contain exactly the same elements.

Examples:

\( C = \{7, 9, -3, 2, 7\} = \{7, 9, -3, 2\} \) (repeat does not count)

\( A = \{9, 2, 7, -3\}, B = \{7, 9, -3, 2\} : A = B \)

\( A = \{\text{dog, cat, horse}\}, B = \{\text{cat, horse, rabbit, dog}\} : A \neq B \)

\( A = \{x : x \in \mathbb{Z} \text{ and } x^2 - x - 2 = 0\}, B = \{2, -1\} : A = B \)

\( (x+1)(x-2) = 0 \)

\( x = 2, -1 \)
Size of Sets

How many elements in a set? (intuitive definition)

Finite set: a set with no elements or its elements can be matched with the elements of some subset \{0,1,2,...,n\} of \(\mathbb{N}\). Informally, a set whose count stop somewhere

Infinite set:

is a set which is not finite

Cardinality of Sets

If a set \(S\) contains \(n\) distinct elements, \(n \in \mathbb{N}\), cardinality of the set \(S\), denoted by \(|S|\), is \(n\).

Examples:

\(A = \{\text{Mercedes, BMW, Porsche}\}, \quad |A| = 3\)
\(B = \{1, \{2, 3\}, \{4, 5\}, 6\}, \quad |B| = ?\)
\(C = \emptyset, \quad |C| = ?\)
\(D = \{x : x \in \mathbb{N} \text{ and } x \leq 7000\}, \quad |D| = ?\)
\(E = \{x \mid x \in \mathbb{N} \text{ and } x \geq 7000\}, \quad |E| = ?\)
\(F = \{\emptyset\}, \quad |F| = ?(F = \{\emptyset\})\)
\(G = \{1,1,1,1,1\}, \quad |G| = ?(G = \{1\})\)
\(H = \{0,1,2,3,...\}, \quad |H| = ?\)
\(I = \{0,1,2,3,...\}, \quad |I| = ?(I = \{H\})\)
Given \( A = \{a_1, \ldots, a_n\} \) if \( a_1 \in B, a_2 \in B, \ldots, a_n \in B \) then \( A \subseteq B \)

A is a **subset** of \( B \), denoted by \( A \subseteq B \), if every element of \( A \) is also an element of \( B \).

\( A = B \implies A \subseteq B \)

A is a **proper subset** of \( B \), denoted by \( A \subset B \), if \( A \subseteq B \) and \( A \neq B \).

\( \emptyset \subseteq A \forall A \) (see text theorem 1)

(But \( \emptyset \in A \) may not hold for any set \( A \)), when \( \emptyset \in A \) true?

\( \emptyset \subseteq A \forall A \) for any set \( A \) (see text theorem 1)

Examples: \( A = \{1, 7, 9, 15\} \) and \( B = \{7, 9\} \) then list all possible subset relations.

- \( B \subseteq A \)
- \( B \subset A \)
- \( A \subseteq A, B \subseteq B \)
- \( \emptyset \subseteq A, \emptyset \subseteq B \)
- \( \emptyset \subseteq A, \emptyset \subseteq B \)

---

**Simple rules/definitions:**

- \( A = B \iff (A \subseteq B) \land (B \subseteq A) \)
- \( (A \subseteq B) \land (B \subseteq C) \implies A \subseteq C \)

Note: \( \land \) “and”

\( \iff \) “if and only if” or “iff”

\( \implies \) “if... then...”

Just use these “translation” for now

Logical connections to be explained later...
**Venn Diagram**

- To illustrate set-theoretic relationship in a diagram
- \( U = \) universal set (set of all elements) that you define (can be N or Z or R or anything!), denoted by a box

Given a set \( A \)

1. \( A \subseteq U \) (a circle in side the U-box)
2. \( A = U \)

Given two sets, \( A \) & \( B \)

1. \( A = \{1,2\}, B = \{3,4\} \)
2. \( A = \{1,2,3\}, B = \{3,4\} \)
3. \( A \subseteq B, (A = \{1,2\}, B = \{1,2,3,4\}) \)
4. \( A = B, (A = \{1,2\}, B = \{1,2\}) \)

---

**Venn Diagram**

- Example:
  
  \( U = \) universal set (set of all elements)
  
  \( A \subseteq B \) and \( B \subseteq C \)
**Power set of S**, denoted by \( P(S) \) or \( 2^S \), contains all possible subsets of S.

\[
\phi \subseteq S \quad \text{and} \quad S \subseteq S
\]

\( \therefore \) \( \emptyset \in P(S) \) and \( S \in P(S) \) are always true.

Example: \( S = \{1, 2, 3\} \)

\[
P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}
\]

\( |P(S)| = 8 \)

Example: \( A = \emptyset \)

\[
P(A) = \{\emptyset\} \quad \text{Note:} \quad |A| = 0, \quad |P(A)| = 1
\]

\[
\emptyset, \{\phi\}, \{\phi, \phi\}, \{\phi\} = \emptyset
\]

- **Cardinality of power sets:** \( |P(A)| = 2^{|A|} \)

Imagine each element in A has an “on/off” switch. Each possible switch configuration in A corresponds to one subset of A, thus one element in \( P(A) \), e.g., \( A = \{x, y, z\} \)
Several Proof Templates

• \( x \in A \): show that \( x \) has the property that defines membership of \( A \).

Example: Let \( A = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \} \), show that \( x=8 \in A \):
- \( x=8 \) is obviously \( x \in \mathbb{N} \) (so the first property of \( A \) holds)
- \( x = 8 = 2k_o \Rightarrow 4 = k_o \)
- Since \( 4 \in \mathbb{N} \) (the second property holds), therefore \( 8 \in A \) Q.E.D.

• \( A \subseteq B \): show that every element of \( A \) is also in \( B \). For \( A \subseteq B \) also show that some element \( x \) of \( B \) is not in \( A \).

Example: Let \( B = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \} \) and 
\( A = \{ n : n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \} \), show that \( A \subseteq B \):
- Let \( x \) be an arbitrary element of \( A \), i.e. \( x = 4t_o + 6 \in A \) (your goal is to show that this property also holds the property of \( B \) too!), first property of \( B \) is obvious.
- second property of \( B \):
- \( x=4t_o + 6 = 2(2t_o+3) = 2k_o \) where \( k_o = 2t_o+3 \)
- since \( t_o \in \mathbb{N} \) it follows that \( k_o \in \mathbb{N} \),
- thus \( x = 2k_o \in B \) is true, therefore \( A \subseteq B \).

• \( A = B \): show that \( A \subseteq B \) and \( B \subseteq A \)

Example: try yourself

• \( A \neq B \): show that \( A \notin B \) (i.e. some element \( x \) of \( A \) is not in \( B \)) or \( B \notin A \) (i.e. some element \( x \) of \( B \) is not in \( A \))

Example: Let \( B = \{ n : n \in \mathbb{N} \text{ and } n = 2k \text{ for some } k \in \mathbb{N} \} \) and \( A = \{ n : n \in \mathbb{N} \text{ and } n = 4t+6 \text{ for some } t \in \mathbb{N} \} \), show that \( B \notin A \):
- Let \( x = 4 = 2*2 \in B \), (Must find an element by yourself)
- Each element of \( A \) must satisfies \( 4t_o + 6 \) for some \( t_o \in \mathbb{N} \). Thus \( A = \{ 6, 10, 14, ... \} \)
- clearly, \( x = 4 \notin A \)
- Therefore, \( B \notin A \)
- Therefore, \( A \neq B \) □

• If \( A \), then \( B \): suppose \( A \) is true then derive \( B \) from the supposition
• If \( A \) if and only if \( B \): show “if \( A \), then \( B \)” and “if \( B \), then \( A \)”

Example: try yourself